

*Journées*

# **ÉQUATIONS AUX DÉRIVÉES PARTIELLES**

Biarritz, 3–7 juin 2012

Albert Mas

**Variational inequalities for singular integral operators**

*J. É. D. P.* (2012), Exposé n° VII, 14 p.

<[http://jedp.cedram.org/item?id=JEDP\\_2012\\_\\_\\_\\_A7\\_0](http://jedp.cedram.org/item?id=JEDP_2012____A7_0)>

cedram

*Article mis en ligne dans le cadre du  
Centre de diffusion des revues académiques de mathématiques  
<http://www.cedram.org/>*

# Variational inequalities for singular integral operators

Albert Mas

## Abstract

In these notes we survey some new results concerning the  $\rho$ -variation for singular integral operators defined on Lipschitz graphs. Moreover, we investigate the relationship between variational inequalities for singular integrals on AD regular measures and geometric properties of these measures. An overview of the main results and applications, as well as some ideas of the proofs, are given.

## 1. Introduction and main results

The topics covered in these notes belong to the area of geometric analysis, which can be considered an interface between harmonic analysis and geometric measure theory. More precisely, they are concerned with the Cauchy and Riesz transforms, two fundamental operators in harmonic analysis, PDE's, and geometric measure theory.

The results presented in these notes have been obtained in a joint work with Xavier Tolsa (see [16], [17], [15]).

### 1.1. Singular integral operators

For the sequel,  $1 \leq n < d$  denote two fixed integers. Given a positive Borel measure  $\mu$  in  $\mathbb{R}^d$ , one way to define the  $n$ -dimensional Riesz transform of  $f \in L^1(\mu)$  is by  $R^\mu f(x) = \lim_{\epsilon \searrow 0} R_\epsilon^\mu f(x)$  (whenever the limit exists), where  $x \in \mathbb{R}^d$  and

$$R_\epsilon^\mu f(x) = \int_{|x-y|>\epsilon} \frac{x-y}{|x-y|^{n+1}} f(y) d\mu(y)$$

denotes the truncation of the Riesz transform at level  $\epsilon > 0$ . When  $d = 2$  (i.e.,  $\mu$  is a Borel measure in  $\mathbb{C}$ ), one defines the Cauchy transform of  $f \in L^1(\mu)$  by

---

The author is partially supported by grants IT-305-07, MTM2010-16232, and 2009SGR-000420.

MSC 2000: 42B20, 42B25.

Keywords:  $\rho$ -variation, singular integral operators, uniform rectifiability.

$C^\mu f(x) = \lim_{\epsilon \searrow 0} C_\epsilon^\mu f(x)$  (whenever the limit exists), where  $x \in \mathbb{C}$  and

$$C_\epsilon^\mu f(x) = \int_{|x-y|>\epsilon} \frac{f(y)}{x-y} d\mu(y)$$

(observe that  $x$ ,  $y$ , and  $f(y)$  are complex numbers). Usually, to avoid the problem of existence of the preceding limits, one considers the associated maximal operators  $R_*^\mu f(x) = \sup_{\epsilon>0} |R_\epsilon^\mu f(x)|$  and  $C_*^\mu f(x) = \sup_{\epsilon>0} |C_\epsilon^\mu f(x)|$ . Notice that the Cauchy transform coincides with the 1-dimensional Riesz transform in the plane, modulo conjugation.

The Cauchy and Riesz transforms are two very important examples of singular integral operators with a Calderón-Zygmund kernel. Namely, given a Borel measure  $\mu$  in  $\mathbb{R}^d$ ,  $\epsilon > 0$ ,  $x \in \mathbb{R}^d$ , and  $f \in L^1(\mu)$ , one considers operators of the form

$$T_\epsilon^\mu f(x) = \int_{|x-y|>\epsilon} K(x-y) f(y) d\mu(y), \quad (1.1)$$

where the kernel  $K : \mathbb{R}^d \setminus \{0\} \rightarrow \mathbb{R}$  satisfies

$$|K(x)| \leq C|x|^{-n}, \quad |\partial_{x_i} K(x)| \leq C|x|^{-n-1}, \quad |\partial_{x_i} \partial_{x_j} K(x)| \leq C|x|^{-n-2}, \quad (1.2)$$

for all  $1 \leq i, j \leq d$  and  $x = (x_1, \dots, x_d) \in \mathbb{R}^d \setminus \{0\}$ , where  $C > 0$  is some constant; and moreover  $K(-x) = -K(x)$  for all  $x \neq 0$  (i.e.  $K$  is odd). The estimate on the second derivatives of  $K$  is not a standard assumption in Calderón-Zygmund theory, but it is a key fact in our results. Notice that the  $n$ -dimensional Riesz transform corresponds to the vector kernel  $(x_1, \dots, x_d)/|x|^{n+1}$ , and the Cauchy transform to  $(x_1, -x_2)/|x|^2$  (so, one may consider  $K$  to be any scalar component of these vector kernels).

## 1.2. Variation operator

The  $\rho$ -variation for martingales and some families of operators has been widely studied in many papers on probability, ergodic theory, and harmonic analysis (see [14], [1], [9], [2], [10], and [25], for example). In these notes we survey some new results concerning the  $\rho$ -variation for families of singular integral operators defined on Lipschitz graphs. By an  $n$ -dimensional Lipschitz graph  $\Gamma \subset \mathbb{R}^d$  we mean any translation and rotation of a set of the type

$$\{x \in \mathbb{R}^d : x = (y, \mathcal{A}(y)), y \in \mathbb{R}^n\},$$

where  $\mathcal{A} : \mathbb{R}^n \rightarrow \mathbb{R}^{d-n}$  is some Lipschitz function with Lipschitz constant  $\text{Lip}(\mathcal{A})$ . We say that  $\text{Lip}(\mathcal{A})$  is the *slope* of  $\Gamma$ .

If  $\mu$  denotes the  $n$ -dimensional Hausdorff measure on an  $n$ -dimensional Lipschitz graph in  $\mathbb{R}^d$ , the  $\rho$ -variation ( $\rho > 2$ ) for the family of operators  $\mathcal{T}^\mu = \{T_\epsilon^\mu\}_{\epsilon>0}$  given in (1.1) is defined by

$$(\mathcal{V}_\rho \circ \mathcal{T}^\mu) f(x) = \sup_{\{\epsilon_m\}} \left( \sum_{m \in \mathbb{Z}} |T_{\epsilon_{m+1}}^\mu f(x) - T_{\epsilon_m}^\mu f(x)|^\rho \right)^{1/\rho}$$

for  $f \in L_{loc}^1(\mu)$  and  $x \in \mathbb{R}^d$ , where the pointwise supremum is taken over all decreasing sequences  $\{\epsilon_m\}_{m \in \mathbb{Z}} \subset (0, \infty)$ . We are also interested in the  $\rho$ -variation for the

family  $\mathcal{T}_\varphi^\mu = \{T_{\varphi_\epsilon}^\mu\}_{\epsilon>0}$ , where

$$T_{\varphi_\epsilon}^\mu f(x) = \int \varphi(|x-y|/\epsilon)K(x-y)f(y)d\mu(y), \quad x \in \mathbb{R}^d,$$

and  $\varphi : [0, \infty) \rightarrow \mathbb{R}$  is a non decreasing function of class  $\mathcal{C}^2$  such that  $\chi_{[2,\infty)} \leq \varphi \leq \chi_{[1,\infty)}$  (the precise value of the constants is not important for our purposes). We usually refer to  $T_\epsilon^\mu$  and  $T_{\varphi_\epsilon}^\mu$  as a *rough* and *smooth truncation*, respectively, of the singular integral with respect to the kernel  $K$  and the measure  $\mu$ .

Our first main result is summarized in the following theorem (see [16], [15]).

**Theorem 1.1.** *Let  $\rho > 2$ . Let  $\mu$  be the  $n$ -dimensional Hausdorff measure restricted to an  $n$ -dimensional Lipschitz graph in  $\mathbb{R}^d$  with slope strictly less than 1. Then, the operator  $\mathcal{V}_\rho \circ \mathcal{T}^\mu$  is bounded in  $L^p(\mu)$  for  $1 < p < \infty$ , from  $L^1(\mu)$  to  $L^{1,\infty}(\mu)$ , and from  $L^\infty(\mu)$  to  $BMO(\mu)$ . The same holds without any restriction on the slope of the Lipschitz graph if one replaces  $\mathcal{T}^\mu$  by  $\mathcal{T}_\varphi^\mu$ .*

The assumption on the slope of the Lipschitz graph is just a technical obstruction due to the methods we use in the proof of the theorem. As we will see,  $\mathcal{V}_\rho \circ \mathcal{T}^\mu$  is actually bounded at least in  $L^2$  for any Lipschitz graph, and even for more general measures (see Theorem 1.5).

Theorem 1.1 applies to the particular cases of the Cauchy and Riesz transforms on Lipschitz graphs. Moreover, it is easy to see that, for some  $C > 0$ ,  $T_*^\mu f \leq C(\mathcal{V}_\rho \circ \mathcal{T}^\mu)f$  for every compactly supported function  $f$ . Thus Theorem 1.1 strengthens the celebrated result of R. Coifman, A. McIntosh, and Y. Meyer about the  $L^2$  boundedness of the Cauchy transform on Lipschitz graphs (the assumption on the slope of the graph can be avoided for this purpose). It is also easily checked that the  $L^p$  boundedness of  $\mathcal{V}_\rho \circ \mathcal{T}^\mu$  yields a new proof of the existence of the principal value  $T^\mu f(x) = \lim_{\epsilon \searrow 0} T_\epsilon^\mu f(x)$  for all  $f \in L^p(\mu)$  and  $\mu$ -almost all  $x \in \mathbb{R}^d$ , without using a dense class of functions in  $L^p(\mu)$ .

Furthermore, from Theorem 1.1 one also gets information on the speed of convergence of the principal value. In fact, the boundedness of the  $\lambda$ -jump operator  $N_\lambda \circ \mathcal{T}^\mu$  and the  $(a, b)$ -upcrossings operator  $N_a^b \circ \mathcal{T}^\mu$  is classically derived from variational inequalities. Given  $\lambda > 0$ ,  $f \in L_{loc}^1(\mu)$  and  $x \in \mathbb{R}^d$ , one defines  $(N_\lambda \circ \mathcal{T}^\mu)f(x)$  as the supremum of all integers  $N$  for which there exist  $0 < \epsilon_1 < \delta_1 \leq \epsilon_2 < \delta_2 \leq \dots \leq \epsilon_N < \delta_N$  so that  $|T_{\epsilon_i}^\mu f(x) - T_{\delta_i}^\mu f(x)| > \lambda$  for each  $i = 1, \dots, N$ . Similarly, given  $a < b$ , one defines  $(N_a^b \circ \mathcal{T}^\mu)f(x)$  to be the supremum of all integers  $N$  for which there exist  $0 < \epsilon_1 < \delta_1 \leq \epsilon_2 < \delta_2 \leq \dots \leq \epsilon_N < \delta_N$  so that  $T_{\epsilon_i}^\mu f(x) < a$  and  $T_{\delta_i}^\mu f(x) > b$  for each  $i = 1, \dots, N$ . Using Theorem 1.1 one obtains the following theorem (see [16], [2]).

**Theorem 1.2.** *Let  $\mu$  be as in Theorem 1.1,  $\rho > 2$ , and  $\lambda > 0$ . For  $1 < p < \infty$ , there exist constants  $C_p, C_1 > 0$  such that*

$$\lambda \|(N_\lambda \circ \mathcal{T}^\mu)f\|^{1/\rho}_{L^p(\mu)} \leq C_p \|f\|_{L^p(\mu)} \quad \text{and}$$

$$\lambda m^{1/\rho} \mu(\{x \in \mathbb{R}^d : (N_\lambda \circ \mathcal{T}^\mu)f(x) > m\}) \leq C_1 \|f\|_{L^1(\mu)}$$

for all  $m \in \mathbb{N}$ . The same holds replacing  $\lambda$  by  $b - a$  and  $N_\lambda$  by  $N_a^b$ , where  $a < b$  are two given real numbers.

These results also hold for the family of smooth truncations  $\mathcal{T}_\varphi^\mu$ .

Concerning the background on the  $\rho$ -variation, a fundamental result is Lépingle's inequality [14], from which the  $L^p$  boundedness of the  $\rho$ -variation for martingales

follows, for  $\rho > 2$  and  $1 < p < \infty$ . From this result on martingales, one deduces that the  $\rho$ -variation for averaging operators (also called differentiation operators) is bounded in  $L^p$ , and similar conclusions hold in the setting of dynamical systems (see [9]). As far as we know, the first work dealing with the  $\rho$ -variation for singular integral operators is the one of J. Campbell, R. L. Jones, K. Reinhold and M. Wierdl ([2]), where the  $L^p$  and weak  $L^1$  boundedness of the  $\rho$ -variation (for  $\rho > 2$ ) for the Hilbert transform was proved. Later on, there appeared other papers showing the  $L^p$  boundedness of the  $\rho$ -variation for singular integrals in  $\mathbb{R}^n$  ([3]), with weights ([7]), and for other operators such as the spherical averaging operator or singular integral operators on parabolas ([10]). Recently, the case of the Carleson operator has been considered too ([12], [25]).

### 1.3. Relationship with uniform rectifiability

For a given measure  $\mu$  in  $\mathbb{R}^d$ , the relationship between the  $L^2(\mu)$  boundedness of singular integrals and the geometric properties of  $\mu$  (such as rectifiability) is an area of research that has attracted much attention in the last years. There are influential contributions, for example, by G. David, P. Jones, P. Mattila, M. Melnikov, T. Murai, S. Semmes, X. Tolsa, J. Verdera, A. Volberg, etc. See [26], for example, for further names and references.

We recall some definitions on geometric measure theory. A Borel measure  $\mu$  in  $\mathbb{R}^d$  is said to be  $n$ -dimensional Ahlfors-David regular, or simply AD regular, if there exists some constant  $C > 0$  such that  $C^{-1}r^n \leq \mu(B(x, r)) \leq Cr^n$  for all  $x \in \text{supp}\mu$  and  $0 < r \leq \text{diam}(\text{supp}\mu)$ . One says that  $\mu$  is  $n$ -rectifiable if there exists a countable family of  $n$ -dimensional  $\mathcal{C}^1$  manifolds  $\{M_i\}_{i \in \mathbb{N}}$  such that  $\mu(\mathbb{R}^d \setminus \bigcup_{i \in \mathbb{N}} M_i) = 0$ . One also says that  $\mu$  is uniformly  $n$ -rectifiable if there exist  $\theta, M > 0$  so that, for each  $x \in \text{supp}\mu$  and  $0 < r \leq \text{diam}(\text{supp}\mu)$ , there is a Lipschitz mapping  $g$  from the  $n$ -dimensional ball  $B^n(0, r) \subset \mathbb{R}^n$  into  $\mathbb{R}^d$  such that  $\text{Lip}(g) \leq M$  and  $\mu(B(x, r) \cap g(B^n(0, r))) \geq \theta r^n$ . The uniform rectifiability is a quantitative stronger version of rectifiability. Thus, in particular, any AD regular uniformly rectifiable measure is actually rectifiable. Notice also that the  $n$ -dimensional Hausdorff measure restricted to an  $n$ -dimensional Lipschitz graph is a uniformly  $n$ -rectifiable measure.

G. David and S. Semmes asked more than twenty years ago the still open question that follows (see, for example, [26, Chapter 7]):

**Question 1.3.** *Is it true that an  $n$ -dimensional AD regular measure  $\mu$  in  $\mathbb{R}^d$  is uniformly  $n$ -rectifiable if and only if  $R_*^\mu$  is bounded in  $L^2(\mu)$ ?*

In [4], G. David and S. Semmes proved the “only if” implication of the question above. Moreover, they gave a positive answer if one replaces, in the question, the  $L^2$  boundedness of  $R_*^\mu$  by the  $L^2$  boundedness of  $T_*^\mu$  for a wide class of odd kernels  $K$ . In this direction, P. Mattila and D. Preiss proved in [22] the following result: *let  $\mu$  be an  $n$ -dimensional AD regular measure in  $\mathbb{R}^d$ . Assume that, for any  $\mathcal{C}^\infty$  radial function  $h : \mathbb{R}^d \setminus \{0\} \rightarrow \mathbb{R}$  such that  $|h(x)| \leq C$  and  $|\nabla h(x)| \leq C|x|^{-1}$  for some fixed constant  $C > 0$ , the operators  $T_\epsilon^\mu$  defined by (1.1) with kernel  $K(x) = h(x)|x|^{-n-1}x$  are bounded in  $L^2(\mu)$  uniformly in  $\epsilon > 0$ . Then,  $\mu$  is  $n$ -rectifiable.*

The “if” implication in the question above was proved by P. Mattila, M. Melnikov and J. Verdera in [21] for the case of the Cauchy transform, that is  $n = 1$  and  $d = 2$ .

Later on, G. David and J. C. Léger proved in [13] that the  $L^2$  boundedness  $C_*^\mu$  implies that  $\mu$  is rectifiable, i.e., they obtained the corresponding “if” implication without the AD regularity assumption (for  $n = 1$  and  $d = 2$ ). Very recently, Question 1.3 has been answered affirmatively by F. Nazarov, X. Tolsa, and A. Volberg for codimension 1, that is, for  $n = d - 1$  ([24]).

When  $\mu$  is the  $n$ -dimensional Hausdorff measure restricted to a set  $E \subset \mathbb{R}^d$  such that  $\mu(E) < \infty$ , the rectifiability of  $\mu$  is also related to the existence of the principal value of the Riesz transform of  $\mu$  for  $\mu$ -a.e.  $x \in E$ . For example, P. Mattila and M. Melnikov showed in [20] that, if  $\mu$  is rectifiable, for all finite Borel measures  $\nu$  there exists  $R^\nu 1(x)$  for  $\mu$ -a.e.  $x \in \mathbb{R}^d$ . In [22], P. Mattila and D. Preiss proved that, under the additional assumption that  $\liminf_{r \rightarrow 0} r^{-n} \mu(B(x, r)) > 0$  for  $\mu$ -a.e.  $x \in E$ , the rectifiability of  $E$  is equivalent to the existence of  $R^\mu 1(x)$   $\mu$ -a.e.  $x \in E$ . Later on, in [31], X. Tolsa removed the assumption on the lower density of  $\mu$ , i.e., he proved that  $\mu$  is rectifiable if and only if the principal value  $R^\mu 1$  exists  $\mu$  almost everywhere. Let us mention that, for the Cauchy transform, the same results were obtained in [19] with some density assumptions, and in [29] by using the notion of curvature of measures. For other results dealing with principal values, Hausdorff measures, rectifiability, and related questions, see also [8], [23], [6], [33], [27], and [28], for example.

The following theorem is our second main result, which might be considered as a partial answer to the question above, and it is proved using Theorem 1.1 (see [17]).

**Theorem 1.4.** *Let  $\rho > 2$ . An  $n$ -dimensional AD regular measure  $\mu$  is uniformly  $n$ -rectifiable if and only if  $\mathcal{V}_\rho \circ \mathcal{R}^\mu$  is bounded in  $L^2(\mu)$ , where  $\mathcal{R}^\mu = \{R_\epsilon^\mu\}_{\epsilon > 0}$ .*

Therefore,  $\mathcal{V}_\rho \circ \mathcal{R}^\mu$  completely characterizes the  $n$ -AD regular measures  $\mu$  which are uniformly rectifiable. Recall that the boundedness of  $\mathcal{V}_\rho \circ \mathcal{R}^\mu$  implies the existence of the principal value  $R^\mu 1$ , which in turn implies rectifiability. Thus our theorem yields stronger conclusions with, a priori, stronger hypotheses. Theorem 1.4 is a direct consequence of the following one (see [17]).

**Theorem 1.5.** *Let  $\mu$  be an  $n$ -dimensional AD regular Borel measure in  $\mathbb{R}^d$ . Then, the following are equivalent:*

- (a)  $\mu$  is uniformly  $n$ -rectifiable,
- (b) for any  $\rho > 2$  and any  $T_\epsilon^\mu$  as in (1.1), the operator  $\mathcal{V}_\rho \circ \mathcal{T}^\mu$  is bounded in  $L^2(\mu)$ ,
- (c) for some  $\rho > 2$ , the operator  $\mathcal{V}_\rho \circ \mathcal{R}^\mu$  is bounded in  $L^2(\mu)$ .

## 1.4. Further results

Denote by  $M(\mathbb{R}^d)$  the space of finite complex Radon measures on  $\mathbb{R}^d$  equipped with the norm given by the variation of measures. The following theorem strengthens the first endpoint estimate of Theorem 1.1 (see [15], [17]).

**Theorem 1.6.** *Let  $\rho > 2$ , and let  $\mu$  be  $n$ -dimensional Hausdorff measure restricted to an  $n$ -dimensional Lipschitz graph with slope strictly less than 1. Then,  $\mathcal{V}_\rho \circ \mathcal{T}$  is a bounded operator from  $M(\mathbb{R}^d)$  to  $L^{1,\infty}(\mu)$ , i.e., there exist a constant  $C > 0$  such*

that, for all  $\lambda > 0$  and all  $\nu \in M(\mathbb{R}^d)$ ,

$$\lambda \mu(\{x \in \mathbb{R}^d : (\mathcal{V}_\rho \circ \mathcal{T})\nu(x) > \lambda\}) \leq C\|\nu\|,$$

where  $\mathcal{T} = \{T_\epsilon\}_{\epsilon>0}$  and

$$T_\epsilon\nu(x) = \int_{|x-y|>\epsilon} K(x-y) d\nu(y). \quad (1.3)$$

In particular,  $\mathcal{V}_\rho \circ \mathcal{T}^\mu$  is of weak type  $(1,1)$ . The same holds without the assumption on the slope of the Lipschitz graph if one replaces  $\mathcal{V}_\rho \circ \mathcal{T}$  by  $\mathcal{V}_\rho \circ \mathcal{T}_\varphi$ .

Denote by  $\mathcal{H}^n$  the  $n$ -dimensional Hausdorff measure in  $\mathbb{R}^d$ . The following corollary is a direct consequence of Theorem 1.6 (see [15]).

**Corollary 1.7.** *Let  $E$  be an  $\mathcal{H}^n$  measurable subset of  $\mathbb{R}^d$  with  $\mathcal{H}^n(E) < \infty$  and such that  $\mathcal{H}^n$  restricted to  $E$  is  $n$ -rectifiable, and let  $K$  be an odd kernel satisfying (1.2). If  $\nu \in M(\mathbb{R}^d)$ , then the principal value  $\lim_{\epsilon \searrow 0} T_\epsilon\nu(x)$  exists for  $\mathcal{H}^n$  almost all  $x \in E$ .*

Given a set  $E \subset \mathbb{R}^d$  as in Corollary 1.7, as far as we know, the existence  $\mathcal{H}^n$  a.e.  $x \in E$  of  $\lim_{\epsilon \searrow 0} T_\epsilon\nu(x)$  for  $\nu \in M(\mathbb{R}^d)$  was already known for odd kernels  $K \in \mathcal{C}^\infty(\mathbb{R}^d \setminus \{0\})$  satisfying

$$|\nabla^j K(x)| \leq C_j |x|^{-n-j} \quad (1.4)$$

for all  $j = 0, 1, 2, 3, \dots$ , or maybe assuming (1.4) only for a finite but big number of  $j$ 's (see [18, Theorems 20.15 and 20.27, Remarks 20.16 and 20.19] and the references therein). However, the result is new if one asks (1.4) only for  $j = 0, 1, 2$ , and so Corollary 1.7 improves on previous results.

Similarly to the  $\rho$ -variation, one may also consider the so-called oscillation operator. Given a decreasing sequence  $\{r_m\}_{m \in \mathbb{Z}} \subset (0, \infty)$ , the oscillation (with respect to  $\{r_m\}_{m \in \mathbb{Z}}$ ) for  $\mathcal{T}^\mu$  is defined by

$$(\mathcal{O} \circ \mathcal{T}^\mu)f(x) = \sup_{\{\epsilon_m\}, \{\delta_m\}} \left( \sum_{m \in \mathbb{Z}} |T_{\epsilon_m}^\mu f(x) - T_{\delta_m}^\mu f(x)|^2 \right)^{1/2}$$

for  $f \in L^1_{loc}(\mu)$  and  $x \in \mathbb{R}^d$ , where the pointwise supremum is taken over all sequences  $\{\epsilon_m\}_{m \in \mathbb{Z}}$  and  $\{\delta_m\}_{m \in \mathbb{Z}}$  such that  $r_{m+1} \leq \epsilon_m \leq \delta_m \leq r_m$  for all  $m \in \mathbb{Z}$ . All the results in these notes also hold replacing  $\mathcal{V}_\rho$  by  $\mathcal{O}$ . Moreover, the norm of the corresponding operators is bounded independently of the sequence that defines  $\mathcal{O}$ .

## 2. On the proof of the main results

For the sake of brevity, and because of its possible applications in boundary value problems on non smooth domains, we mainly focus our attention only on the proof of Theorem 1.1. Theorem 1.2 follows from Theorem 1.1 by standard arguments (see [2], for example). At the end of these notes we make some comments concerning Theorems 1.5 and 1.6.

### 2.1. The $\alpha$ and $\beta$ coefficients

For the proof of Theorem 1.1, it is a key fact to develop a multiscale analysis on the underlying measure using the so-called  $\alpha$  and  $\beta$  coefficients.

Given  $m \in \mathbb{N}$ ,  $\lambda > 0$ , and a cube  $Q \subset \mathbb{R}^m$  (i.e.  $Q = [0, b]^m + a$  with  $a \in \mathbb{R}^m$  and  $b > 0$ ), let  $\ell(Q)$  denote the side length of  $Q$ , let  $z_Q$  denote the center of  $Q$  and let  $\lambda Q$  be the cube with center  $z_Q$  and side length  $\lambda\ell(Q)$ .

Let  $\mu$  be a locally finite Borel measure in  $\mathbb{R}^d$ . Given  $1 \leq p < \infty$  and a cube  $Q \subset \mathbb{R}^d$ , one sets (see [5])

$$\beta_{p,\mu}(Q) = \inf_L \left\{ \frac{1}{\ell(Q)^n} \int_{2Q} \left( \frac{\text{dist}(y, L)}{\ell(Q)} \right)^p d\mu(y) \right\}^{1/p}, \quad (2.1)$$

where the infimum is taken over all  $n$ -planes  $L$  in  $\mathbb{R}^d$ . For  $p = \infty$  one replaces the  $L^p$  norm by the supremum norm, that is,

$$\beta_{\infty,\mu}(Q) = \inf_L \left\{ \sup_{y \in \text{supp}\mu \cap 2Q} \frac{\text{dist}(y, L)}{\ell(Q)} \right\}, \quad (2.2)$$

where the infimum is taken over all  $n$ -planes  $L$  in  $\mathbb{R}^d$  again. These coefficients were introduced by P. W. Jones in [11] for  $p = \infty$  and by G. David and S. Semmes in [4] for  $1 \leq p < \infty$ .

Let  $F \subset \mathbb{R}^d$  be the closure of an open set. Given two finite Borel measures  $\sigma, \nu$  in  $\mathbb{R}^d$ , one sets

$$\text{dist}_F(\sigma, \nu) = \sup \left\{ \left| \int f d\sigma - \int f d\nu \right| : \text{Lip}(f) \leq 1, \text{supp} f \subset F \right\}. \quad (2.3)$$

It is easy to check that this is a distance in the space of finite Borel measures  $\sigma$  such that  $\text{supp}\sigma \subset F$  and  $\sigma(\partial F) = 0$ . Moreover, it turns out that this distance is a variant of the well known Wasserstein distance  $W_1$  from optimal transportation (see [34, Chapter 1]). See [18, Chapter 14] for other properties of  $\text{dist}_F$ .

Given a cube  $Q \subset \mathbb{R}^d$  which intersects  $\text{supp}\mu$ , set  $B_Q = B(z_Q, 6\sqrt{d}\ell(Q))$ . Then one defines (see [32])

$$\alpha_\mu(Q) = \frac{1}{\ell(Q)^{n+1}} \inf_{c \geq 0, L} \text{dist}_{B_Q}(\mu, c\mathcal{H}_L^n), \quad (2.4)$$

where the infimum is taken over all constants  $c \geq 0$  and all  $n$ -planes  $L$  in  $\mathbb{R}^d$ , and where  $\mathcal{H}_L^n$  denotes the  $n$ -dimensional Hausdorff measure restricted to  $L$ . For convenience, if  $Q$  does not intersect  $\text{supp}\mu$ , one sets  $\alpha_\mu(Q) = 0$ .

The following result characterizes uniform rectifiability in terms of the  $\alpha$  and  $\beta$  coefficients.

**Theorem 2.1.** *Let  $\mu$  be an  $n$ -dimensional AD regular measure in  $\mathbb{R}^d$ , and consider any  $p \in [1, 2]$ . Then, the following are equivalent:*

- (a)  $\mu$  is uniformly  $n$ -rectifiable.
- (b) There exists  $C > 0$  such that, for any cube  $R \subset \mathbb{R}^d$ ,

$$\sum_{Q \in \mathcal{D}(R)} \beta_{p,\mu}(Q)^2 \mu(Q) \leq C\mu(R),$$

where  $\mathcal{D}(R)$  stands for the collection of cubes in  $\mathbb{R}^d$  contained in  $R$  which are obtained by splitting  $R$  dyadically.

(c) There exists  $C > 0$  such that, for any cube  $R \subset \mathbb{R}^d$ ,

$$\sum_{Q \in \mathcal{D}(R)} \alpha_\mu(Q)^2 \mu(Q) \leq C \mu(R).$$

The equivalence (a) $\iff$ (b) in Theorem 2.1 was proved by G. David and S. Semmes in [4], and the equivalence (a) $\iff$ (c) was proved by X. Tolsa in [32].

## 2.2. Martingales

The first step in the proof of Theorem 1.1 is to relate the variation for singular integral operators to the variation for martingales, and to use the known results on the variation for martingales. Recall a particular case of Lépingle's inequality (see [14] or [10]):

**Theorem 2.2.** *Let  $(X, \Sigma, \lambda)$  be a  $\sigma$ -finite measure space and  $\rho > 2$ . There exists  $C > 0$  such that  $\|\mathcal{V}_\rho(\mathcal{G})\|_{L^2(\lambda)} \leq C \|\mathcal{G}\|_{L^2(\lambda)}$  for every martingale  $\mathcal{G} = \{G_m\}_{m \in \mathbb{Z}} \in L^2(\lambda)$ , where  $\|\mathcal{G}\|_{L^2(\lambda)} = \sup_{m \in \mathbb{Z}} \|G_m\|_{L^2(\lambda)}$ ,*

$$\mathcal{V}_\rho(\mathcal{G})(x) = \sup_{\{\epsilon_m\}} \left( \sum_{m \in \mathbb{Z}} |G_{\epsilon_{m+1}}(x) - G_{\epsilon_m}(x)|^\rho \right)^{1/\rho}$$

and the supremum is taken over all increasing sequences  $\{\epsilon_m\}_{m \in \mathbb{Z}} \subset \mathbb{Z}$ .

We are going to introduce a suitable martingale for our purposes. For the rest of this section, let  $\mu$  be as in Theorem 1.1, and assume that  $\Gamma = \{x \in \mathbb{R}^d : x = (y, \mathcal{A}(y)), y \in \mathbb{R}^n\}$  is the corresponding Lipschitz graph. We may also assume that  $\mathcal{A}$  has compact support. Given  $m \in \mathbb{Z}$  and  $a \in \mathbb{R}^n$ , we set

$$\widetilde{D}_m^a = a + [0, 2^{-m}]^n \subset \mathbb{R}^n \quad \text{and} \quad D_m^a = \widetilde{D}_m^a \times \mathbb{R}^{d-n} \subset \mathbb{R}^d.$$

Set  $\mathcal{D}_m^a = \{D_m^{a+2^{-m}k} \subset \mathbb{R}^d : k \in \mathbb{Z}^n\}$  (for a fixed  $a$ , the projection of  $\bigcup_{m \in \mathbb{Z}} \mathcal{D}_m^a$  onto  $\mathbb{R}^n$  is a translation of the standard dyadic lattice in  $\mathbb{R}^n$ ). Notice that, since  $\Gamma$  is an  $n$ -dimensional Lipschitz graph,  $\mu(D_m^a)$  is comparable to  $2^{-mn}$  for all  $m \in \mathbb{Z}$ ,  $a \in \mathbb{R}^n$ . For  $D \in \mathcal{D}_m^a$  and  $x \in D$ , we set

$$E_D \mu(x) = \frac{1}{\mu(D)} \int_D \int_{D^c} K(z-y) d\mu(y) d\mu(z).$$

Finally, given  $a \in \mathbb{R}^n$ , we define the martingale

$$E_m^a \mu(x) = \sum_{D \in \mathcal{D}_m^a} \chi_D(x) E_D \mu(x)$$

for  $x \in \mathbb{R}^d$  and  $m \in \mathbb{Z}$ , where  $\chi_D$  denotes the characteristic function of  $D$ .

Let us make some comments to understand better the nature of  $E_m^a \mu$ . Roughly speaking, if we forget the truncations, we have

$$\int_D \int_D K(z-y) d\mu(y) d\mu(z) = 0$$

because of the antisymmetry of  $K$  (use Fubini's theorem). Hence, if we set  $T^\mu f(z) = \int K(z-y)f(y) d\mu(y)$  for  $f \in L^1(\mu)$ , then  $\int_D T^\mu \chi_D d\mu = 0$  and

$$E_m^a \mu(x) = \frac{1}{\mu(D)} \int_D T^\mu \chi_{D^c} d\mu = \frac{1}{\mu(D)} \int_D T^\mu 1 d\mu$$

for  $x \in D \in \mathcal{D}_m^a$ . Therefore,  $E_m^a \mu(x)$  is the average of the function  $T^\mu 1$  on the set  $D \in \mathcal{D}_m^a$  which contains  $x$ . So, it is clear that, for a fixed  $a \in \mathbb{R}^n$ ,  $\{E_m^a \mu\}_{m \in \mathbb{Z}}$  is a martingale. In [23] it is shown that  $\{E_m^a \mu\}_{m \in \mathbb{Z}}$  is actually well defined and a martingale. Finally, for  $x \in \mathbb{R}^d$ , we define

$$E_m \mu(x) = 2^{mn} \int_{\{a \in \mathbb{R}^n : x \in D_m^a\}} E_m^a \mu(x) d\mathcal{L}^n a$$

where  $\mathcal{L}^n$  denotes the Lebesgue measure in  $\mathbb{R}^n$  (notice that  $\mathcal{L}^n(\{a : x \in D_m^a\}) = 2^{-mn}$ ). Thus,  $E_m \mu$  is an average (of the  $m$ 'th term) of some martingales depending on a parameter  $a \in \mathbb{R}^n$ . Set  $E\mu = \{E_m \mu\}_{m \in \mathbb{Z}}$ . Using Theorem 2.2, in [16] we prove the following theorem, which can be considered the starting point to prove Theorem 1.1.

**Theorem 2.3.** *Fix a dyadic cube  $\tilde{P} \subset \mathbb{R}^n$ , set  $P = \tilde{P} \times \mathbb{R}^{d-n}$ , and assume that  $\mathcal{A}$  is supported in  $\tilde{P}$ . Let  $\rho > 2$ . There exists  $C > 0$  independent of  $P$  such that  $\|\mathcal{V}_\rho(E\mu)\|_{L^2(\mu)}^2 \leq C\mu(P)$ .*

### 2.3. Sketch of the proof of Theorem 1.1

As pointed out above, the proof relies on two basic facts: the known  $L^2$  boundedness of the  $\rho$ -variation for martingales explained in section 2.2 and the good geometric properties of Lipschitz graphs from a measure theoretic point of view.

The starting point of the proof is Theorem 2.3, where the  $L^2$  boundedness of the  $\rho$ -variation (of a convex combination) of some particular martingales is stated. So, the next step consists in relating the results on martingales of Theorem 2.3 with the  $\rho$ -variation for singular integrals on Lipschitz graphs, and this is the aim of the following proposition (see [16]). We denote by  $\mathcal{D}$  the standard dyadic lattice in  $\mathbb{R}^d$ .

**Proposition 2.4.** *Let  $\mathcal{A}$  and  $\mu$  be as in Theorem 2.3. For each  $x \in \text{supp}\mu$ , define*

$$W\mu(x)^2 = \sum_{j \in \mathbb{Z}} |T_{\varphi_{2^{-j}}}^\mu 1(x) - E_j \mu(x)|^2,$$

$$S\mu(x)^2 = \sup_{\{\epsilon_m\}} \sum_{j \in \mathbb{Z}} \sum_{m \in \mathbb{Z}: \epsilon_m, \epsilon_{m+1} \in [2^{-j-1}, 2^{-j})} |T_{\varphi_{\epsilon_{m+1}}}^\mu 1(x) - T_{\varphi_{\epsilon_m}}^\mu 1(x)|^2,$$

where the supremum is taken over all decreasing sequences of positive numbers  $\{\epsilon_m\}_{m \in \mathbb{Z}}$ . Then, there exist  $C_1, C_2 > 0$  such that

$$\|W\mu\|_{L^2(\mu)}^2 + \|S\mu\|_{L^2(\mu)}^2 \leq C_1 \sum_{Q \in \mathcal{D}} (\alpha_\mu(Q)^2 + \beta_{2,\mu}(Q)^2) \mu(Q) \leq C_2 \mu(P).$$

The last inequality in Proposition 2.4 can be easily derived from the packing condition that the  $\alpha$ 's and  $\beta$ 's satisfy (i.e., Theorem 2.1(b) and (c)).

The  $\alpha$  and  $\beta$  coefficients are two fundamental tools in the study of  $W\mu$  and  $S\mu$ , which are used to measure the flatness of the graph  $\Gamma$  at different scales in order to estimate the terms which appear in the sums defining  $W\mu$  and  $S\mu$ . To use the  $\alpha$  coefficients to relate the  $\rho$ -variation for martingales with the  $\rho$ -variation for singular integrals, it is a key fact that we consider a family of smooth truncations and an average of martingales instead of rough truncations and a fixed martingale, because the  $\alpha$ 's are defined in terms of Lipschitz functions (see section 2.4 for more details, where a concrete example is shown).

Combining Proposition 2.4 with the  $L^2$  estimates for the  $\rho$ -variation on the average of martingales  $E\mu$  in Theorem 2.3, we obtain local  $L^2$  estimates of  $\mathcal{V}_\rho \circ \mathcal{T}_\varphi^\mu$  when  $\Gamma$  is any Lipschitz graph (the restriction  $\text{Lip}(\mathcal{A}) < 1$  is not necessary). More precisely, we separate the sum in the definition of  $\mathcal{V}_\rho \circ \mathcal{T}_\varphi^\mu$  into two parts, which are classically called *short* and *long variation*. The short variation corresponds to the sum  $S\mu$  in Proposition 2.4, where the indices run over those  $m \in \mathbb{Z}$  such that both  $\epsilon_m$  and  $\epsilon_{m+1}$  lie in the same dyadic interval, and can be handled using the  $\alpha$ 's and  $\beta$ 's. The long variation corresponds to the sum over the indices  $m \in \mathbb{Z}$  such that  $\epsilon_m$  and  $\epsilon_{m+1}$  lie in different dyadic intervals, so one may assume that the  $\epsilon_m$ 's are dyadic numbers, i.e.,  $\epsilon_m = 2^{-j}$  for some  $j \in \mathbb{Z}$ . It is handled by comparing  $T_{\varphi_{2^{-j}}}^\mu 1$  with  $E_j \mu$ , and then using Theorem 2.4 and the fact the  $\rho$ -variation for  $E\mu$  is bounded in  $L^2(\mu)$ , by Theorem 2.3.

Using the local  $L^2$  estimates for  $\mathcal{V}_\rho \circ \mathcal{T}_\varphi^\mu$ , combined with rather standard techniques in Calderón-Zygmund theory, we obtain the endpoint estimates of Theorem 1.1 for  $\mathcal{V}_\rho \circ \mathcal{T}_\varphi^\mu$ . Then, by interpolation, we obtain the  $L^p$  boundedness of this operator in the whole range  $1 < p < \infty$ , and in particular the  $L^2$  boundedness.

The next step is to replace the family of smooth truncations  $\varphi$  by the rough one. We obtain the  $L^2$  boundedness of  $\mathcal{V}_\rho \circ \mathcal{T}^\mu$  by comparing this operator with  $\mathcal{V}_\rho \circ \mathcal{T}_\varphi^\mu$ , and by estimating the difference in terms of the  $\alpha$  and  $\beta$  coefficients, decomposing a function  $f \in L^2(\mu)$  using a suitable wavelet basis. It is in this step where we need the assumption  $\text{Lip}(\mathcal{A}) < 1$ . Roughly speaking, when dealing with rough truncations, we need the estimate

$$\mu(A(x, a, b)) \leq C(b - a)b^{n-1} \quad \text{for all } x \in \text{supp}\mu, \quad (2.5)$$

where  $A(x, a, b) = \{y \in \mathbb{R}^d : a \leq |y - x| \leq b\}$  (that is the case if, for example,  $\mathcal{A}$  is affine), but this estimate may fail if  $\text{Lip}(\mathcal{A}) \geq 1$ .

Finally, by adapting the proof of [3, Theorem B] to our setting and using standard techniques in Calderón-Zygmund theory, we show that the  $L^2$  boundedness of  $\mathcal{V}_\rho \circ \mathcal{T}^\mu$  yields the endpoint estimates of Theorem 1.1, and we obtain the  $L^p$  boundedness in the whole range  $1 < p < \infty$  by interpolation again. This finishes the proof.

## 2.4. How the $\alpha$ and $\beta$ coefficients come into play

This section is devoted to illustrate how the  $\alpha$  and  $\beta$  coefficients appear when studying variational inequalities for singular integral operators. We only present an example for the  $\alpha$ 's because for the  $\beta$ 's the arguments are similar.

We intend to estimate one term of the sum defining  $S\mu$  in Proposition 2.4 (say  $|T_{\varphi_{\epsilon_{m+1}}}^\mu 1(x) - T_{\varphi_{\epsilon_m}}^\mu 1(x)|$ , for some  $\epsilon_m$  and  $\epsilon_{m+1}$ ) by means of the  $\alpha$  coefficients. To simplify and facilitate the exposition, we may assume that we are in the most favorable situation. That is, assume that  $x \in Q \cap \text{supp}\mu$  for some cube  $Q \in \mathcal{D}$  with  $\ell(Q) = 2^{-j}$  for some  $j \in \mathbb{Z}$ , that  $\epsilon_m = \frac{9}{10}2^{-j}$  and  $\epsilon_{m+1} = 2^{-j-1}$ , that  $x$  belongs to the  $n$ -plane  $L$  which minimizes  $\alpha_\mu(Q)$ , and that the constant  $c$  which minimizes  $\alpha_\mu(Q)$  is equal to 1. We want to estimate

$$\begin{aligned} & \left| T_{\varphi_{2^{-j-1}}}^\mu 1(x) - T_{\varphi_{2^{-j}9/10}}^\mu 1(x) \right| \\ &= \left| \int \left( \varphi(|x - y|2^{j+1}) - \varphi(|x - y|2^j 10/9) \right) K(x - y) d\mu(y) \right|. \end{aligned}$$

Set  $\psi(y) = \varphi(|x-y|2^{j+1}) - \varphi(|x-y|2^j 10/9)$ , so  $\psi$  is supported in  $B_Q$  and  $0 \leq \psi \leq 1$ . Moreover,  $|x-y|$  is comparable to  $2^{-j} = \ell(Q)$  for all  $y \in \text{supp}\psi$ . Since  $\varphi$  is smooth, it is easy to check that there exists  $C > 0$  depending only on  $\varphi$  such that  $|\nabla\psi| \leq C\ell(Q)^{-1}$ . Combining these estimates with (1.2) we deduce that  $\Psi(y) = \psi(y)K(x-y)$  is supported in  $B_Q$ , and that for some fixed  $C > 0$  we have  $|\nabla\Psi| \leq C\ell(Q)^{-n-1}$ .

Finally, since  $\psi$  is a radial function (with center  $x$ ),  $K$  is odd, and  $L$  is a plane which contains  $x$ , we obviously have  $\int \Psi d\mathcal{H}_L^n = 0$ . Therefore, for some  $C > 0$  depending only on  $\varphi$ , we conclude

$$\begin{aligned} \left| T_{\varphi_{2^{-j-1}}}^\mu 1(x) - T_{\varphi_{2^{-j} 10/9}}^\mu 1(x) \right| &= \left| \int \Psi d\mu \right| = \left| \int \Psi d\mu - \int \Psi d\mathcal{H}_L^n \right| \\ &\leq \frac{C}{\ell(Q)^{n+1}} \text{dist}_{B_Q}(\mu, c\mathcal{H}_L^n) = C \alpha_\mu(Q), \end{aligned}$$

and we are done.

## 2.5. Further comments

In this section we only give an overview of the main ingredients for proving Theorems 1.5 and 1.6 (see [17] and [15] for the details).

Concerning Theorem 1.5, the implication (b)  $\implies$  (c) is obvious. The proof of (a)  $\implies$  (c) can be separated in three main steps. For the first one, we assume that  $\mu$  is the  $n$ -dimensional Hausdorff measure restricted to an  $n$ -dimensional Lipschitz graph. Then, an application of Theorem 1.1 gives that  $\mathcal{V}_\rho \circ \mathcal{T}_\varphi^\mu$  is bounded in  $L^2(\mu)$ . For the second step, we use a *good  $\lambda$  inequality* to derive, from the  $L^2$  boundedness of  $\mathcal{V}_\rho \circ \mathcal{T}_\varphi^\mu$  when  $\mu$  is the Hausdorff measure on a Lipschitz graph, the  $L^2$  boundedness of  $\mathcal{V}_\rho \circ \mathcal{T}_\varphi^\mu$  when  $\mu$  is a more general measure whose support contains *big pieces of Lipschitz graphs*. Applying that method once again we obtain the  $L^2$  boundedness of  $\mathcal{V}_\rho \circ \mathcal{T}_\varphi^\mu$  when  $\mu$  is a measure whose support contains *big pieces of sets which contain big pieces of Lipschitz graphs*, which in turn is equivalent to say that  $\mu$  is uniformly rectifiable (see [5] for the precise definitions). In order to run the good  $\lambda$  method, we need the following estimate:

$$\left| (\mathcal{V}_\rho \circ \mathcal{T}_\varphi^\mu) f \chi_{(2B)^c}(x) - (\mathcal{V}_\rho \circ \mathcal{T}_\varphi^\mu) f \chi_{(2B)^c}(z) \right| \leq CM^\mu f(x)$$

for  $x, z \in B$ , where  $B$  is any ball,  $f \in L^1(\mu)$ , and  $M^\mu$  denotes the Hardy-Littlewood maximal operator with respect to  $\mu$ . The estimate may fail for rough truncations, i.e., for  $\mathcal{V}_\rho \circ \mathcal{T}^\mu$ , so we need to use smooth truncations. The third and last step consists on obtaining the  $L^2$  boundedness of  $\mathcal{V}_\rho \circ \mathcal{T}^\mu$  on uniformly rectifiable measures  $\mu$ , and this is a combination of two ingredients: a decomposition of the support of  $\mu$  using a *corona decomposition* in the sense of [5], which organizes the good geometric/measure theoretic information of  $\mu$ , and the comparison of  $\mathcal{V}_\rho \circ \mathcal{T}^\mu$  with the smooth version  $\mathcal{V}_\rho \circ \mathcal{T}_\varphi^\mu$ , which we already know that is bounded in  $L^2(\mu)$ , using a suitable Haar basis adapted to the corona decomposition of  $\mu$ .

For the proof of (c)  $\implies$  (a), one first notices that the  $L^2$  boundedness of  $\mathcal{V}_\rho \circ \mathcal{R}^\mu$  implies the  $L^2$  boundedness of its smooth version, say  $\mathcal{V}_\rho \circ \mathcal{R}_\varphi^\mu$ . In [32] it is shown that if

$$\sum_{j \in \mathbb{Z}} \left\| R_{\varphi_{2^{-j-1}}}^\mu \chi_Q - R_{\varphi_{2^{-j}}}^\mu \chi_Q \right\|_{L^2(\mu)}^2 \leq C\mu(Q)$$

for all  $Q \in \mathcal{D}$  then  $\mu$  is uniformly rectifiable, which in turn is a consequence of the fact that

$$\sum_{P \in \mathcal{D}(Q)} \beta_{2,\mu}(P)^2 \mu(P) \leq C \sum_{j \in \mathbb{Z}} \left\| R_{\varphi_{2^{-j-1}}}^\mu \chi_{3Q} - R_{\varphi_{2^{-j}}}^\mu \chi_{3Q} \right\|_{L^2(\mu)}^2 + C\mu(Q)$$

and Theorem 2.1. Notice that

$$\sum_{j \in \mathbb{Z}} \left\| R_{\varphi_{2^{-j-1}}}^\mu \chi_Q - R_{\varphi_{2^{-j}}}^\mu \chi_Q \right\|_{L^2(\mu)}^2 \leq \left\| (\mathcal{V}_2 \circ \mathcal{R}_\varphi^\mu) \chi_Q \right\|_{L^2(\mu)}^2,$$

thus if  $\mathcal{V}_2 \circ \mathcal{R}_\varphi^\mu$  is bounded in  $L^2(\mu)$ , we are done. However, the 2-variation operator is unbounded in general (even for the case of martingales). For  $\rho > 2$ , we use these ideas combined with a deep result in [5] called the *weak geometric lemma* and a stopping time argument.

The proof of Theorem 1.6 is based on a nontrivial modification of the proof of [3, Theorem B] using a Calderón-Zygmund decomposition for general measures developed in [30]. The estimate (2.5) is necessary in our arguments, so we must require  $\text{Lip}(\mathcal{A}) < 1$  in the statement of the theorem.

## References

- [1] J. Bourgain, *Pointwise ergodic theorems for arithmetic sets*, Inst. Hautes Études Sci. Publ. Math. 69 (1989), pp. 5–45.
- [2] J. Campbell, R. L. Jones, K. Reinhold, and M. Wierdl, *Oscillation and variation for the Hilbert transform*, Duke Math. J. 105 (2000), pp. 59–83.
- [3] J. Campbell, R. L. Jones, K. Reinhold, and M. Wierdl, *Oscillation and variation for singular integrals in higher dimensions*, Trans. Amer. Math. Soc. 35 (2003), pp. 2115–2137.
- [4] G. David and S. Semmes, *Singular integrals and rectifiable sets in  $R^n$ : au-delà des graphes lipschitziens*, Astérisque No. 193 (1991).
- [5] G. David and S. Semmes, *Analysis of and on uniformly rectifiable sets*, Mathematical Surveys and Monographs, 38, American Mathematical Society, Providence, RI (1993).
- [6] H. Farag, *The Riesz kernels do not give rise to higher-dimensional analogues of the Menger-Melnikov curvature*, Pub. Mat. 43 (1999), no. 1, pp. 251–260.
- [7] T. A. Gillespie and J. L. Torrea, *Dimension free estimates for the oscillation of Riesz transforms*, Israel Journal of Math. 141 (2004), pp. 125–144.
- [8] P. Huovinen, *Singular integrals and rectifiability of measures in the plane*, dissertation, University of Jyväskylä, Jyväskylä (1997). Ann. Acad. Sci. Fenn. Math. Diss. No. 109 (1997), 63 pp.
- [9] R. L. Jones, R. Kaufman, J. Rosenblatt, and M. Wierdl, *Oscillation in ergodic theory*, Ergodic Theory and Dynam. Sys. 18 (1998), pp. 889–936.

- [10] R. L. Jones, A. Seeger, and J. Wright, *Strong variational and jump inequalities in harmonic analysis*, Trans. Amer. Math. Soc. 360 (2008), pp. 6711–6742.
- [11] P. W. Jones, *Square functions, Cauchy integrals, analytic capacity, and harmonic measure*, Harmonic analysis and partial differential equations (El Escorial, 1987), Lecture Notes in Math., 1384, Springer, Berlin, 1989, pp. 24–68.
- [12] M. Lacey and E. Terwilleger, *A Wiener-Wintner theorem for the Hilbert transform*, Ark. Mat. 46 (2008), 2, pp. 315–336.
- [13] J. C. Léger, *Menger curvature and rectifiability*, Ann. of Math., 149 (1999), pp. 831–869.
- [14] D. Lépingle, *La variation d'ordre  $p$  des semi-martingales*, Z. Wahrscheinlichkeitstheorie Verw. Gebiete 36 (1976), pp. 295–316.
- [15] A. Mas, *Variation for singular integrals on Lipschitz graphs:  $L^p$  and endpoint estimates*, to appear in Trans. Amer. Math. Soc. (2012).
- [16] A. Mas and X. Tolsa, *Variation and oscillation for singular integrals with odd kernel on Lipschitz graphs*, to appear in Proc. London Math. Soc. (2012).
- [17] A. Mas and X. Tolsa, *Variation for the Riesz transform and uniform rectifiability*, to appear in J. Eur. Math. Soc. (2012).
- [18] P. Mattila, *Geometry of sets and measures in Euclidean spaces*, Cambridge Stud. Adv. Math. 44, Cambridge Univ. Press, Cambridge (1995).
- [19] P. Mattila, *Cauchy singular integrals and rectifiability of measures in the plane*, Adv. Math. 115 (1995), pp. 1–34.
- [20] P. Mattila and M. S. Melnikov, *Existence and weak-type inequalities for Cauchy integrals of general measures on rectifiable curves and sets*, Proc. Amer. Math. Soc. 120 (1994), no. 1, pp. 143–149.
- [21] P. Mattila, M. S. Melnikov and J. Verdera, *The Cauchy integral, analytic capacity, and uniform rectifiability*, Ann. of Math. (2) 144 (1996), pp. 127–136.
- [22] P. Mattila and D. Preiss, *Rectifiable measures in  $\mathbb{R}^n$  and existence of principal values for singular integrals*, J. London Math. Soc. (2) 52 (1995), no. 3, pp. 482–496.
- [23] P. Mattila and J. Verdera *Convergence of singular integrals with general measures*, J. Eur. Math. Soc. 11 (2009), pp. 257–271.
- [24] F. Nazarov, X. Tolsa, and A. Volberg. Private communication. In preparation.

- [25] R. Oberlin, A. Seeger, T. Tao, C. Thiele, and J. Wright *A variation norm Carleson theorem*, J. Eur. Math. Soc. 14 (2012), 2, pp. 421–464.
- [26] H. Pajot, *Analytic capacity, rectifiability, Menger curvature and the Cauchy integral*, Lecture Notes in Math. 1799, Springer (2002).
- [27] A. Ruiz de Villa and X. Tolsa, *Non existence of principal values of signed Riesz transforms of non integer dimension*, Indiana Univ. Math. J. 59 (2010), no. 1, pp. 115–130.
- [28] X. Tolsa. *Cotlar’s inequality without the doubling condition and existence of principal values for the Cauchy integral of measures*, J. Reine Angew. Math. 502 (1998), pp. 199–235.
- [29] X. Tolsa. *Principal values for the Cauchy integral and rectifiability*, Proc. Amer. Math. Soc. 128(7) (2000), pp. 2111–2119.
- [30] X. Tolsa. *A proof of the weak  $(1, 1)$  inequality for singular integrals with non doubling measures based on a Calderón-Zygmund decomposition*, Pub. Mat. 45(1) (2001), pp. 163–174.
- [31] X. Tolsa, *Principal values for Riesz transforms and rectifiability*, J. Funct. Anal., vol. 254(7) (2008), pp. 1811–1863.
- [32] X. Tolsa, *Uniform rectifiability, Calderón-Zygmund operators with odd kernel, and quasiorthogonality*, Proc. London Math. Soc. 98(2) (2009), pp. 393–426.
- [33] M. Vihtilä, *The boundedness of Riesz  $s$ -transforms of measures in  $\mathbb{R}^n$* , Proc. Amer. Math. Soc. 124 (1996), no. 12, pp 3739–3804.
- [34] C. Villani, *Topics in optimal transportation*, Graduate Studies in Mathematics, vol. 58 (2003). American Math. Soc., Providence RI.

DEPARTAMENTO DE MATEMÁTICAS,  
 UNIVERSIDAD DEL PAÍS VASCO –  
 EUSKAL HERRIKO UNIBERTSITATEA,  
 48080 LEIOA, BIZKAIA (SPAIN)  
 amasblesa@gmail.com