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Frederic Rousset

Abstract

The aim of this note is to present recent results obtained with N. Masmoudi [29] on the free surface Navier-Stokes equation with small viscosity.

1. Introduction

We are interested in the motion of a viscous incompressible fluid with a free surface under the influence of gravity. The equations of motion read:

$$\partial_t u + u \cdot \nabla u + \nabla p = \varepsilon \Delta u, \quad \nabla \cdot u = 0, \quad x \in \Omega_t, \tag{1.1}$$

where $u \in \mathbb{R}^3$ is the velocity of the fluid and $p \in \mathbb{R}$ is the apparent pressure, $p = p^F + p^h$ with p^F the pressure of the fluid and $p^h = gx_3$ the hydrostatic pressure. We assume that the fluid domain is the simplest one:

$$\Omega_t = \Big\{ x \in \mathbb{R}^3, \quad x_3 < h(t, x_1, x_2) \Big\}$$

with $h(t, x_1, x_2)$ which defines the free surface is also an unkown in the problem.

The boundary conditions on the free surface $x_3 = h(t, x_1, x_2)$ are the following:

$$\partial_t h = u \cdot N = -u_1 \partial_1 h - u_2 \partial_2 h + u_3, \quad (x_1, x_2) \in \mathbb{R}^2 \tag{1.2}$$

where N is the outward normal given by $N = (-\partial_1 h, \, -\partial_2 h, \, 1)^t$ and

$$p\, N - 2\, \varepsilon\, Su\, N = g\, h\, N \tag{1.3}$$

where

$$Su = \frac{1}{2}\Big(\nabla u + \nabla u^t \Big).$$

The first boundary condition is of kinematic nature, it basically states that the normal speed of the interface, is given by the normal velocity of the fluid. The second boundary condition is of physical nature, it means that one can impose the normal component of the stress tensor (we neglect surface tension) on the free surface.

We are interested in the motion of the fluid at large Reynolds number, this is the reason for the small parameter $\varepsilon > 0$ in the equation (1.1).

In the limit ε tends to zero, we expect the solution of (1.1) to converge towards a solution of the free surface Euler equation. Indeed, it is a natural conjecture in fluid mechanics that the physical solutions of the Euler equations are the ones that can be obtained by vanishing viscosity limit from the Navier-Stokes equation. In order to perform rigorously this justification, we want to:

- Get the existence of a strong solution on an interval of time $[0, T]$ independent of ε
- Get uniform estimates sufficient to pass to the limit towards a solution of the Euler equation and thus recover the well-posedness of the free surface Euler equation.

There are two main difficulties in order to implement this strategy for the Navier-Stokes equation with free surface boundary conditions. The first one is related to the control of the regularity of the surface uniformly in ε and the second one is related to the presence of a boundary layer in the vicinity of the free surface. Note that for such an approach to be valid we need to get a functional space in which both the Navier-Stokes and the inviscid, Euler, equations are well posed.

2. Boundary layers

We shall first discuss the problem of boundary layers. For the Navier-Stokes equation, even with boundary conditions on a rigid wall, it is well known that the standard local existence results of strong solutions are valid on an interval of time $[0, T^\varepsilon]$ with T^ε that tends to zero when ε goes to zero and thus they cannot be used in order to pass to the limit from strong compactness arguments. Note that even in the two-dimensional case where strong solutions are known to be global, the uniform estimates are also only valid on an interval of time that vanishes when ε goes to zero. We also point out that the way to justify the inviscid limit from Leray weak solutions by using weak compactness arguments is also unknown. This is due to the poor information that we get from the energy dissipation inequality:

$$\frac{d}{dt}\frac{1}{2}\|u^\varepsilon\|^2 + \varepsilon\|\nabla u^\varepsilon(t)\|^2 \leq 0$$

where $\|\cdot\|$ stands for the L^2 norm. All this difficulties are due to the presence of a boundary layer that is to say a small region close to the bondary where the gradient of the solution is very large.

In the case where one imposes a Dirichlet boundary condition on a rigid wall $u^\varepsilon_{/z=0} = 0$ in the simplest domain $\Omega = \{z > 0\}$, the expected description of the solution is:

$$u^\varepsilon \sim u^E + V(t, y, z/\sqrt{\varepsilon})$$

where u^E is a solution of the Euler equation and $V(t, y, Z)$, the boundary layer, is supposed to be fastly decreasing in its last variable. One immediately see that u^ε cannot be bounded in H^s, $s > 5/2$ which is the standard space in which the 3-D Euler equation is well-posed. Nevertheless, one can try to justify rigorously the above asymptotic expansion i.e. to write the solution under the form

$$u^\varepsilon = u^E + V(t, y, z/\sqrt{\varepsilon}) + r^\varepsilon \tag{2.1}$$

and study the equation for the remainder r^ε in order to prove that it goes to zero (of course if needed one can start from an approximate solution with more terms). There are many difficulties in the case of Dirichlet boundary conditions:

- the profile V solves the Prandtl equation which is often ill-posed for non analytic data: [12].

- even when one can construct it, the approximate solution can be unstable [15], [20]

Therefore, for the Navier-Stokes equation with Dirichlet boundary condition, the justification of the inviscid limit is known only in the analytic framework [32].

Nevertheless, we point out that the above approach is efficient even for general quasilinear hyperbolic-parabolic systems (thus also for compressible fluids and MHD equations) when the boundary is non-characteristic (this happens for example with injection or succion boundary conditions), in this case the size of the boundary layer is ε (in the ansatz (2.1), V depends on z/ε) or in dimension one. We refer to [14, 16, 17, 30, 19, 31, 36].

A more favorable boundary condition on a rigid wall for which the boundary layer is similar to the one about a free surface is the Navier (slip) boundary condition which reads

$$u \cdot N = 0, \, \Pi S u N = \alpha \Pi u, \quad \Pi = Id - \frac{N \otimes N}{|N|^2} \tag{2.2}$$

where $\alpha \geq 0$ is a fixed parameter. The justification of the inviscid limit for the Navier-Stokes equation with Navier boundary condition has been studied for a long time, [3], [10], [24], [22]. In particular, in the three-dimensional case, in [22], it is proven by a modulated energy type approach that for a sufficiently smooth solution of the Euler equation defined on some interval $[0, T]$, an L^2 convergence holds on $[0, T]$. Nevertheless, these results, in particular the last one in 3D do not provide uniform estimates in strong norms. In the case of the Navier boundary condition, this is not needed in order to pass to the limit since one can start from a Leray global weak solution but since the existence of weak solutions is not known for the Navier-Stokes equation with a free surface, in order to see the problem with Navier boundary condition as a model problem for the free surface, we need to prove that a strong solution in a suitable functional space of the Navier-Stokes equation exists on an interval of time independent of ε. For some special type of Navier boundary conditions or boundaries, some uniform H^3 (or $W^{2,p}$, with p large enough) estimates and a uniform time of existence for Navier-Stokes when the viscosity goes to zero have been recently obtained (see [37, 7, 6]). For these special boundary conditions, the main part of the boundary layer vanishes which allows this uniform control in some limited regularity Sobolev space. Nevertheless, as shown in [23], in the case of Navier boundary conditions, the asymptotic expansion is under the form

$$u^\varepsilon = u^E + \sqrt{\varepsilon} V(t, y, z/\sqrt{\varepsilon}) + \varepsilon r^\varepsilon \tag{2.3}$$

and the profile V except for exceptional boundary conditions (i.e. for some choice of α) is not zero. With this expansion, we see that u^ε still cannot be bounded in H^s $s > 5/2$ when V is not zero. Nevertheless, these case seems much more favorable since one can expect the Lipschitz norm of u^ε to be uniformly bounded. Consequently, it seems reasonable to get uniform estimates by using the Sobolev conormal spaces that are classically used in the study of hyperbolic initial boundary value problems [4, 18, 21, 35].

We shall use the following definition:

In $\mathcal{S}$, defined by $x = (y, z)$, $y \in \mathbb{R}^2$, $z < 0$. Let us introduce the vector fields

$$Z_i = \partial_i,\ i = 1,\ 2, \quad Z_3 = \frac{z}{1-z}\partial_z.$$

We define the Sobolev conormal spaces H^m_{co} on L^2 as

$$H^m_{co}(\mathcal{S}) = \Big\{ f \in L^2(\mathcal{S}), \quad Z^\alpha f \in L^2(\mathcal{S}), \quad |\alpha| \leq m \Big\}$$

$$\|f\|_m^2 = \sum_{|\alpha| \leq m} \|Z^\alpha f\|^2_{L^2}.$$

We can also consider the Sobolev conormal spaces built on L^∞:

$$W^{m,\infty}_{co}(\mathcal{S}) = \Big\{ f \in L^\infty(\mathcal{S}), \quad Z^\alpha f \in L^\infty(\mathcal{S}), \quad |\alpha| \leq m \Big\}$$

$$\|f\|_{m,\infty} = \sum_{|\alpha| \leq k} \|Z^\alpha f\|_{L^\infty}.$$

For general domains with smooth boundaries, the spaces can be defined by using local charts.

In [28], we have obtained:

Theorem 2.1. *For $m \geq m_0$, and Ω a smooth domain, consider u_0 a divergence free vector field with zero normal component on the boundary and such that $u_0 \in H^m_{co}$, $\nabla u_0 \in H^{m-1}_{co}$ and $\nabla u_0 \in W^{1,\infty}_{co}$. Then , there exists $T > 0$ such that for every $\varepsilon \in (0,1)$, there is a unique solution u^ε of the Navier-Stokes equation (1.1) with Navier boundary condition with initial data u_0. Moreover, we have the uniform estimates:*

$$\sup_{[0,T]} \Big(\|u(t)\|_m + \|\nabla u(t)\|_{m-1} + \|\nabla u(t)\|_{1,\infty} \Big) + \varepsilon \int_0^T \|\nabla^2 u(s)\|^2_{m-1}\, ds \leq C.$$

From the above uniform estimates, it is easy to get:

Corollary 2.2. *u^ε converges strongly towards u solution of the Euler equation and such that*

$$\sup_{[0,T]} \Big(\|u(t)\|_m + \|\nabla u(t)\|_{m-1} + \|\nabla u(t)\|_{1,\infty} \Big) < +\infty$$

The proof of this result is based on conormal energy estimates of u and its normal derivative and on direct L^∞ type estimates for ∇u. These L^∞ estimates which are the most delicate to get are obtained directly from the equation and not from Sobolev embedding. Indeed, in view of the behaviour (2.3), one cannot get uniform estimates for $\|\partial_z u\|_{L^\infty}$ from Sobolev embedding results.

3. The free surface Navier-Stokes and Euler equations

Local existence results for the free surface Navier-Stokes equation (1.1), (1.2), (1.3) are now classical [5], [34]. The unknown domain is flattened by using Lagrangian coordinates and the local existence result is obtained in "parabolic" Sobolev spaces $\mathcal{H}^r([0,T] \times \Omega_0) = H^{\frac{r}{2}}(0,T,L^2) \cap L^2([0,T], H^r(\Omega_0))$, $r > 3$. The rough idea is that since the change of variable to Lagrangian coordinates is under the form $X = \mathrm{Id} + \mathcal{O}(T)$, one can write the equation (1.1) in the fixed domain Ω_0 under the form

$$\partial_t v + \nabla p - \varepsilon \Delta v = \cdots, \quad \nabla \cdot v = \cdots$$

where $\cdots$ in the first equation contains in particular the convection term and also terms under the form $\mathcal{O}(T)\partial_{ij}^2 v$ which come from the change of coordinates in the Laplacian. The crucial step to get a local existence result through a fixed point argument is thus to get maximal regularity estimates for the Stokes problem in the initial domain Ω_0. The smoothing effect on the velocity for the Navier-Stokes equation makes the regularity of the surface rather easy to control.

In the case of the Euler equation with a free surface, namely

$$\partial_t u + u \cdot \nabla u + \nabla p = 0, \quad \nabla \cdot u = 0, \quad x \in \Omega_t \tag{3.1}$$

with the boundary condition (1.2) and

$$p = gh$$

on the boundary, local existence results have been obtained only recently. The difficulty is that once the problem is reset in a fixed domain, the new velocity v and the surface h are at the same level of regularity. Note that the problem is well-posed only if the Taylor sign condition

$$-\partial_N p + g \geq c_0 > 0 \tag{3.2}$$

is verified. Under this condition, the first local existence result in H^s for s sufficiently large has been obtained in a series of paper [9, 26, 27]. It is based on the reformulation of the equation in Lagrangian coordinates and the use of the Nash-Moser iteration scheme. More recent results have been obtained by using other approaches [11, 33]. Note that much more can be said when u is assumed in addition to be irrotationnal (we obtain the famous water-waves system), we refer for example to [38], [25], [39], [13] and the talk by Nicolas Burq. Nevertheless, note that irrotational solutions are not interesting for our problem since in the context of the Navier-Stokes equation, vorticity on the boundary is automatically created.

4. Main result

We shall now describe our approach to get an existence result which is uniform with respect to ε for (1.1), (1.2), (1.3). Note that in order to have uniform estimates, we shall need to assume a Taylor sign condition (3.2). We also point out that as in the case of the Navier boundary conditions, we cannot get uniform H^s estimates due to the presence of boundary layers and we shall thus use Sobolev conormal spaces. In order to state a result, we need to chose a way to fix the domain. Many choices are possible, we shall use a smoothing diffeomorphism defined by

$$\Phi(t, \cdot) : \, x = (y, z), \, z < 0 \mapsto (y, \varphi(t, y, z) = Az + \eta(t, y, z))$$

with η defined through its Fourier transform by

$$\mathcal{F}_y \eta = \chi(|\xi| z) \hat{h}$$

where χ is a smooth compactly supported function which takes the value one in the vicinity of zero. The number $A > 0$ is chosen in order to have $\partial_z \varphi \geq 1$ at the initial time which ensures that Φ is a diffeomorphism.

The main advantage of this choice is that η has a standard Sobolev regularity while for other choices like Lagrangian coordinates where Φ is directly attached to the velocity, Φ will only get from the velocity a Sobolev conormal regularity. This creates some additionnal difficulties in places. With this choice, one easily gets for η the following type of estimates

Proposition 4.1. *We have the following estimates for* η

$$\forall s \geq 0, \quad \|\nabla \eta(t)\|_{H^s(\mathcal{S})} \leq C_s |h(t)|_{s+\frac{1}{2}},$$
$$\forall s \in \mathbb{N}, \quad \|\eta\|_{W^{s,\infty}} \leq C_s |h|_{s,\infty},$$

For functions defined on the boundary, the norms $|\cdot|_s$ and $|\cdot|_{s,\infty}$ refer to the standard Sobolev norms.

Next, we set $v = u \circ \Phi$, $q = p \circ \Phi$. This yields an equation for (v, q, η) in the fixed domain $\mathcal{S} = \{x = (y, z),\ z < 0\}$

$$\partial_t^\varphi v + \Big(v \cdot \nabla^\varphi\Big) v + \nabla^\varphi q = \varepsilon \Delta^\varphi v, \quad \nabla^\varphi \cdot v = 0, \quad x \in \mathcal{S} \tag{4.1}$$

where the new differential operators are defined by

$$\partial_i^\varphi = \partial_i - \frac{\partial_i \varphi}{\partial_z \varphi} \partial_z, \quad i = 0,\, 1,\, 2, \quad \partial_3^\varphi = \partial_z^\varphi = \frac{1}{\partial_z \varphi} \partial_z$$

and the gradient ∇^φ and Laplacian Δ^φ are defined in a natural way by using these operators. On the boundary, we obtain

$$\partial_t h = v \cdot N, \quad q\, N - 2\, \varepsilon\, S^\varphi v\, N = g\, h\, N, \quad z = 0. \tag{4.2}$$

Before stating our main result, we also need to define precisely the form of the Taylor sign condition that we shall use. By using the divergence free condition, we get as usual that the pressure q solves the elliptic equation

$$\Delta^\varphi q = -\nabla^\varphi \cdot (v \cdot \nabla^\varphi v).$$

Moreover, by using the second boundary condition, we get that on the boundary

$$q_{/z=0} = 2 \varepsilon S^\varphi v\, n \cdot n + g h,$$

where n is the unitary outward normal to Ω_t. We shall thus decompose the pressure into an "Euler" part and a "Navier-Stokes" part by setting $q = q^E + q^{NS}$ with

$$\Delta^\varphi q^E = -\nabla^\varphi \cdot (v \cdot \nabla^\varphi v), \quad q^E_{/z=0} = g h$$

and

$$\Delta^\varphi q^{NS} = 0, \quad q^{NS}_{/z=0} = 2 \varepsilon S^\varphi v\, n \cdot n.$$

The main idea is that the part q^{NS} which is small can be always controlled by using the energy dissipation of the Navier-Stokes equation while q^E which is of order one is the part which should converge to the pressure of the Euler equation when ε goes to zero. Consequently, the Taylor sign condition has to be imposed on q^E. After the change of coordinates, this becomes

$$g - \partial_z^\varphi q^E_{/z=0} \geq c_0 > 0. \tag{4.3}$$

Our main result reads.

Theorem 4.2. *For $m \geq 6$, assuming that the above Rayleigh condition is matched at $t = 0$, then for sufficiently smooth initial data, there exists $T > 0$ and $C > 0$ independent of ε such that the solution of* (4.1), (4.2) *satisfies :*

$$\sup_{[0,T]} \Big(\|v\|_m^2 + |h|_m^2 + \|\partial_z v\|_{m-2}^2 + \|\nabla v\|_{1,\infty}^2 \Big) + \|\partial_z v\|^2_{L^4([0,T], H^{m-1}_{co})} \leq C.$$

Moreover, we also have the estimates

$$\sup_{[0,T]}\Big(\varepsilon|h|^2_{m+\frac{1}{2}}+\varepsilon\|\partial_{zz}v\|^2_{L^\infty}\Big)+\varepsilon\int_0^T\Big(\|\nabla v\|^2_m+\|\nabla\partial_z v\|^2_{m-2}\Big)\leq C$$

Note that the first estimate in the above result is weaker than in Theorem 2.1. since we have a control of $\|\partial_z v\|_{H^{m-1}_{co}}$ which is only L^4 in time and not L^∞. This is is linked to the regularity of the pressure in our problem as we shall see below.

By using the above uniform estimates, one can justify the inviscid limit from standard (strong) compactness arguments. Note that the above result does not rely on the construction of an asymptotic expansion under the form (2.3), thus we do not use the a priori knowledge that the Euler equation is well-posed in Sobolev spaces. Consequently, we get the local well-posedness of the free surface Euler equation (in conormal Sobolev spaces) as a corollary.

The complete proof of this result can be found in [29]. The aim of the next section is to describe the main steps of the proof.

5. Sketch of the proof

Since local existence results are classical for the Navier-Stokes equation, the main difficulty is to prove that the solution can be continued on an interval of time independent of ε. We thus need to prove that the quantities that appear in the statement of Theorem 4.2 can be controlled on an interval of time independent of ε. We can get an estimate in closed form through four steps. Note that in the following, we shall work on an interval of time for which we assume that the Taylor sign condition is verified and the map $\Phi(t,\dot{)}$ is indeed a diffeomorphism.

Step 1: Estimates of v and h

The starting point is the energy identity for the system which reads:

Proposition 5.1. *For any smooth solution, we have the energy identity:*

$$\frac{d}{dt}\Big(\int_{\mathcal{S}}|v|^2\,d\mathcal{V}_t+g\int_{z=0}|h|^2\,dy\Big)+4\varepsilon\int_{\mathcal{S}}|S^\varphi v|^2\,d\mathcal{V}_t=0.$$

Here $d\mathcal{V}_t$ stands for the natural volume element induced by the change of variable (4.1): $d\mathcal{V}_t=\partial_z\varphi(t,y,z)\,dydz$.

Proof

: By using standard integration by parts and the divergence free condition, we first get that

$$\begin{aligned}&\frac{d}{dt}\int_{\mathcal{S}}|v|^2\,d\mathcal{V}_t+4\varepsilon\int_{\mathcal{S}}|S^\varphi v|^2\,d\mathcal{V}_t\\&=2\int_{z=0}\Big(2\varepsilon S^\varphi v-q\mathrm{Id}\Big)N\cdot v\,dy.\end{aligned}$$

By using successively the two boundary conditions (4.2), we obtain

$$2\int_{z=0}\Big(2\varepsilon S^\varphi v-q\mathrm{Id}\Big)N\cdot v\,dy=-2\int_{z=0}gh\,v\cdot N\,dy=-\int_{z=0}g\frac{d}{dt}|h|^2\,dy$$

and the result follows.

The next step is to estimates higher order conormal derivatives: we want to estimate $Z^\alpha v$ and $Z^\alpha h$ for $1 \leq |\alpha| \leq m$. The difficulty here is that the coefficients in the equation (4.1) are not smooth enough (even with the use of the smoothing diffeomorphism that we have taken) to neglect the commutators in an usual way. For example, for the transport terms which reads,

$$\partial_t^\varphi + v \cdot \nabla^\varphi = \partial_t + v_y \partial_y + \frac{1}{\partial_z \varphi}\Big(v \cdot N - \partial_t \eta)\partial_z, \quad N = (-\partial_1 \varphi, -\partial_2 \varphi, 1)^t$$

the commutator between Z^α and this term in the equation involves in particular the term $(v \cdot Z^\alpha N)\partial_z v$ which can be estimated only with the help of $\|Z^\alpha N\| \sim |h|_{m+\frac{1}{2}}$. This yields a loss of 1/2 derivative. We also get similar problems when we compute for the pressure term the commutator between Z^α and $\nabla^\varphi q$. The way to solve this difficulty was pointed out by Alinhac in [2], one can use the good unknown $V^\alpha = Z^\alpha v - \partial_z^\varphi v Z^\alpha \eta$. Indeed, let us set

$$\mathcal{N}(v, q, \varphi) = \partial_t^\varphi v + \Big(v \cdot \nabla^\varphi\Big)v + \nabla^\varphi q - 2\varepsilon \nabla^\varphi \cdot \Big(S^\varphi v\Big).$$

Then, if $\mathcal{N}(v, q, \varphi) = 0$, the linearized equation can be written under the form

$$\begin{aligned} &D\mathcal{N}(v, q, \varphi) \cdot (\dot{v}, \dot{q}, \dot{\varphi}) = \\ &\Big(\partial_t^\varphi + (v \cdot \nabla^\varphi) - 2\varepsilon \nabla^\varphi \cdot \Big(S^\varphi \cdot\Big)\Big)\Big(\dot{v} - \partial_z^\varphi v\, \dot{\varphi}\Big) + \nabla^\varphi\Big(\dot{q} - \partial_z^\varphi q\, \dot{\varphi}\Big) \\ &+\Big(\dot{v} \cdot \nabla^\varphi\Big)v - \dot{\varphi}(\partial_z^\varphi v \cdot \nabla^\varphi)v. \end{aligned}$$

This means that the fully linearized equation has the same structure as the equation linearized with respect to the v variable only thanks to the introduction of the good unknown.

By using this crucial remark, we get that the equation for $(Z^\alpha v, Z^\alpha q, Z^\alpha \eta)$ can be written as

$$\partial_t^\varphi V^\alpha + v \cdot \nabla^\varphi V^\alpha + \nabla^\varphi Q^\alpha - 2\varepsilon \nabla^\varphi \cdot S^\varphi V^\alpha = l.o.t.$$

with $V^\alpha = Z^\alpha v - \partial_z^\varphi v Z^\alpha \eta$, $Q^\alpha = Z^\alpha q - \partial_z^\varphi q Z^\alpha \eta$ and hence we can perform an L^2 type energy estimate for this equation. Let us just explain where the Taylor sign condition occurs in this step. By using standard energy estimates, we have in particular

$$\frac{d}{dt}\frac{1}{2}\int_{\mathcal{S}} |V^\alpha|^2 \, d\mathcal{V}_t + \int_{\mathcal{S}} \nabla^\varphi(Q^{E,\alpha}) \cdot V^\alpha \, d\mathcal{V}_t = \cdots$$

where $Q^{E,\alpha} = Z^\alpha q^E - \partial_z^\varphi q^E Z^\alpha \eta$. For the pressure term, we can write

$$\begin{aligned} \int_{\mathcal{S}} \nabla^\varphi(Q^{E,\alpha}) \cdot V^\alpha \, d\mathcal{V}_t &= \int_{z=0} (Z^\alpha q^E - \partial_z^\varphi q^E Z^\alpha h)\, V^\alpha \cdot N \, dy \\ &= \int_{z=0} (g Z^\alpha h - \partial_z^\varphi q^E Z^\alpha h)\, V^\alpha \cdot N \, dy + \cdots \\ &= \int_{z=0} (g Z^\alpha h - \partial_z^\varphi q^E Z^\alpha h) \cdot \partial_t Z^\alpha h + \cdots \\ &= \frac{d}{dt}\frac{1}{2}\int_{z=0} (g - \partial_z^\varphi q^E)|Z^\alpha h|^2 + \cdots \end{aligned}$$

In the two first lines, we have used successively the fact that $p^E = gh$ on the boundary and the first part of the boundary condition (4.2). Therefore, we have a good control of the regularity of the surface only if the sign condition $g - \partial_z^\varphi q^E \geq c_0 > 0$ is matched.

The main conclusion of this step will be that

$$\left\| \left(Z^m v - \partial_z^\varphi v Z^m \eta \right)(t) \right\|^2 + |h(t)|_m^2 \leq C_0 + t\Lambda(R) + \int_0^t \|\partial_z v\|_{m-1}^2$$

where C_0 depends only on the initial data as soon as

$$Q_m(t) = \|v\|_m^2 + |h|_m^2 + \|\partial_z v\|_{m-2}^2 + \|v\|_{2,\infty}^2 + \|\partial_z v\|_{1,\infty}^2 + \varepsilon \|\partial_{zz} v\|_{L^\infty}^2 \leq R$$

for $t \in [0, T^\varepsilon]$.

Step 2: Normal derivative estimates I

In order to close the argument, we need to have estimates on $\partial_z v$. We shall first estimate $\|\partial_z v\|_{L_t^\infty(H_{co}^{m-2})}$. This is not sufficient to control the right hand side in the above estimate, but this will be important in order to get L^∞ estimates. The main idea is to use the equivalent quantity

$$S_N = \Pi S^\varphi v \, N$$

which vanishes on the boundary. This allows to perform conormal estimates on the convection-diffusion type equation with homogeneous Dirichlet boundary condition satisfied by S_N. This yields again an estimate under the form

$$\|\partial_z v(t)\|_{m-2}^2 \leq C_0 + t\Lambda(R) + \int_0^t \|\partial_z v\|_{m-1}^2.$$

Step 3: L^∞ estimates

We also have to estimate the L^∞ norms that occur in the definition of Q_m. The estimate of $\|v\|_{2,\infty}$ is a consequence of the anisotropic Sobolev estimate:

$$\|f\|_{2,\infty}^2 \lesssim \|\partial_z f\|_{k-2} \, \|f\|_k, \quad k \geq 5.$$

Consequently, the difficult part is to estimate $\|\partial_z v\|_{1,\infty}$. Again, it is more convenient to estimate the equivalent quantity $\|S_N\|_{1,\infty}$ since S_N solves a convection diffusion equation with homogeneous boundary condition. The estimate of $\|S_N\|_{L^\infty}$ is a consequence of the maximum principle for this equation. The estimates for $\|Z_i S_N\|_{L^\infty}$ are more difficult to obtain. The main reason is that a crude estimate of the commutator between Z_i and the variable coefficient operator Δ^φ involves terms with two normal derivatives of S_N and hence three normal derivatives of v. To fix this difficulty, we note that at this step, the regularity of the surface is not really a problem: we want to estimate a fix low number of derivatives of v in L^∞ while m can be considered as large as we need. Consequently, the idea is to change the coordinate system into a normal geodesic one in order to get the simplest possible expression for the Laplacian. By neglecting all the terms that can be estimated by the previous steps, we get a simple one-dimensional equation under the form

$$\partial_t \tilde{S}_N + z\partial_z w_3(t, y, 0)\partial_z \tilde{S}_N + w_h(t, y, 0) \cdot \nabla_h \tilde{S}_N - \varepsilon \partial_{zz} \tilde{S}_N = l.o.t$$

where $\tilde{S}_N$ stands for S_N expressed in the new coordinate system and w is the vector field that we obtain from v by the change of variable. This is a one-dimensional Fokker Planck type equation for which the Green function is explicit and hence, we can use it to estimate $\|Z_i \tilde{S}_N\|_{L^\infty}$.

Again the conclusion of this step is an estimate under the form

$$\|\partial_z v\|_{1,\infty}^2 + \varepsilon \|\partial_{zz} v\|_{L^\infty}^2 \leq C_0 + t\Lambda(R) + \int_0^t \|\partial_z v\|_{m-1}^2$$

.

Step 4: Normal derivative estimate II

In order to close our estimate, we still need to estimate $\|\partial_z v\|_{m-1}$. For this estimate it does not seem a good idea to use S_N as an equivalent quantity for $\partial_z v$. Indeed, the equation for $Z^{m-1} S_N$ involves $Z^{m-1} D^2 p$ as a source term and we note that since the Euler part of the pressure involves an harmonic function that verifies $p^E = gh$ on the boundary, we have that

$$Z^{m-1} D^2 p^E \sim Z^{m-1} D^{\frac{3}{2}} h \sim |h|_{m+\frac{1}{2}}$$

and hence we do not have enough regularity of the surface. For a better treatment of the pressure, it is natural to try to use the vorticity $\omega = \nabla^\varphi \times v$ in place since we have the equation.

$$\partial_t Z^{m-1} \omega + V \cdot \nabla Z^{m-1} \omega - \varepsilon \Delta^\varphi Z^{m-1} \omega = l.o.t$$

Nevertheless, note that while for the Euler equation the vorticity which solves a transport equation with characteristic boundary is very easy to estimate, for the Navier-Stokes equation in domain with boundaries it is much more difficult. The difficulty in the case of the Navier-Stokes equation is that we need an estimate of the value of the vorticity on the boundary to estimate it in the interior. Since on the boundary we have roughly $Z^{m-1}\omega \sim Z^m v + Z^m h$, we only have by using a trace estimate a (uniform) control by known quantities (and in particular the energy dissipation of the Navier-Stokes equation) of

$$\sqrt{\varepsilon} \int_0^t |Z^{m-1} \omega_{/z=0}|_{L^2(\mathbb{R}^2)}^2.$$

To guess what is the best estimate that we can expect, we can study a similar situation for the heat equation

$$\partial_t f - \varepsilon \Delta f = 0,\ z < 0, \quad f_{/z=0} = f^b$$

where we assume that the boundary value f^b is such that

$$\sqrt{\varepsilon} \int_0^T \int_{\mathbb{R}^2} |f^b(t,y)|^2 \, dt dy \leq C.$$

By using a Laplace-Fourier transform, we get that

$$\hat{f} = e^{\left(\gamma + i\tau + \varepsilon |\xi|^2\right)^{\frac{1}{2}} \frac{z}{\sqrt{\varepsilon}}} \hat{f}^b, \quad z < 0$$

and hence we get

$$|\hat{f}(\gamma, \tau, \xi, \cdot)|_{L^2_z}^2 \leq \frac{\sqrt{\varepsilon}}{(\gamma + |\tau| + \varepsilon |\xi|^2)^{\frac{1}{2}}} |\hat{f}^b|^2.$$

This yields

$$\int_0^{+\infty} e^{-2\gamma t} \left\| (\gamma + |\partial_t|)^{\frac{1}{4}}) f \right\|^2 dt \leq \sqrt{\varepsilon} \int_0^{+\infty} e^{-2\gamma t} \|f^b\|^2 \, dt.$$

Consequently, we see that we get a control of f in $H^{\frac{1}{4}}((0,T),L^2)$ which gives by Sobolev embedding an estimate of f in $L^4([0,T],L^2(\Omega))$ only.

Motivated by this computation on the heat equation, we shall get an estimate of $\|Z^{m-1}\omega\|_{L^4((0,T),L^2)}$ by using a microlocal energy estimate. Note that the transport term in the equation has an important effect. Indeed, in the previous example of the heat equation, if we add a constant drift $c\cdot\nabla f$ in the equation, we obtain a smoothing effect under the form

$$\int_0^{+\infty} e^{-2\gamma t}\Big\|(\gamma+|\partial_t+c\cdot\nabla|)^{\frac{1}{4}})f\Big\|^2\,dt.$$

Consequently, we first switch into Lagrangian coordinates in order to eliminate the transport term and we look for an estimate of $\|(Z^{m-1}\omega)\circ X\|_{H^{\frac{1}{4}}([0,T],L^2)}$. For this estimate, we use a microlocal symmetrizer based on a "partially" semiclassical paradifferential calculus i.e. based on the weight $(\gamma^2+|\tau|^2+|\sqrt{\varepsilon}\,\xi|^4)^{\frac{1}{4}})$. The main properties of this calculus can be seen as a consequence of the general quasihomogeneous calculus studied in [30].

This finally allows to get an estimate of $\|Z^{m-1}\partial_z v\|_{L^4((0,T),L^2)}$.

The general estimate follows by combining the estimates of the four steps. Note that in the end, we also have to check that the Taylor sign condition and the condition that $\Phi(t,\cdot)$ is a diffeomorphism remain true.

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