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Abstract

The purpose of this talk is to present some recent results about the Cauchy theory of the gravity water waves equations (without surface tension). In particular, we clarify the theory as well in terms of regularity indexes for the initial conditions as fin terms of smoothness of the bottom of the domain (namely no regularity assumption is assumed on the bottom). Our main result is that, following the approach developed in [1, 2], after suitable para-linearizations, the system can be arranged into an explicit symmetric system of quasilinear waves equation type, and consequently can be solved at the usual levels of regularity (initial data in $H^s, s > 1+d/2$). In particular, the system can be solved for initial surfaces having undounded curvature. As another illustration of this reduction, we show that in fact following the analysis by Bahouri-Chemin and Tataru for quasi-linear wave equations, using Strichartz estimates, the regularity threshold can be further lowered, which allows to obtain well posedness for non lipschitz initial velocity fields. We also take benefit from our low regularity result and an elementary (though seemingly yet unknown) observation to solve a question raised by Boussinesq on the water-wave system in a canal.

1. The equations

We are interested in the study of the Cauchy problem for the water waves system in arbitrary dimension. Water waves are waves on the free surface of a fluid (think of the interface between air and water for the oceans, lakes, canals and swimming pools...). Here we consider an incompressible inviscid liquid, having unit density, occupying a domain with a free surface, of the form

$$\Omega = \{ (t,x,y) \in [0,T] \times \mathbf{R}^d \times \mathbf{R} \,:\, (x,y) \in \Omega(t) \},$$

where $\Omega(t)$ is a domain comprised between the free surface

$$\Sigma_t = \{(x,y); y = \eta(t,x)\}$$

and the bottom Γ. More precisely, starting from a fixed connected domain $\mathcal{O} \subset \mathbb{R}^d$ containing a strip of positive length, we define

$$\Omega_h(t) := \Big\{(x,y) \in \mathbf{R}^d \times \mathbf{R} \,:\, \eta(t,x) - h < y < \eta(t,x)\Big\} \subset \mathcal{O}, \tag{1.1}$$

and we assume that the domain $\Omega(t)$ has the form

$$\Omega(t) = \{(x,y) \in \mathcal{O}; y < \eta(t,x)\},$$

and

$$\Omega_h \subset \Omega(t) \tag{1.2}$$

We will denote by Σ the free surface

$$\Sigma = \{(t,x,y); y = \eta(t,x)\},$$

and by $\Gamma = \partial\Omega \setminus \Sigma$ the bottom .

Remark 1.1. (i) Notice that no regularity assumption is made on the bottom.

(ii) Our method applies in the case where the bottom is time dependent (with the additional assumption in this case that the bottom is Lipschitz).

Hereafter, $d \geq 1$, t denotes the time variable and x and y denote the horizontal and vertical spatial variables. Below we use the following notations

$$\nabla = (\partial_{x_i})_{1\leq i\leq d}, \quad \nabla_{x,y} = (\nabla, \partial_y), \quad \Delta = \sum_{1\leq i\leq d} \partial_{x_i}^2, \quad \Delta_{x,y} = \Delta + \partial_y^2.$$

We thus consider the following system:

$$\begin{cases} \partial_t v + v \cdot \nabla_{x,y} v + \nabla_{x,y} P = -g e_y, \text{ in } [0,T] \times \Omega \\ \partial_t \eta = \sqrt{1 + |\nabla\eta|^2} v \cdot n, \text{ on } [0,T] \times \Sigma \\ v \cdot n \mid_\Gamma = 0, \qquad P \mid_\Sigma = 0, \\ \operatorname{div}(v) = 0, \qquad \operatorname{curl}(v) = 0 \end{cases} \tag{1.3}$$

This system describes a flow whose eulerian velocity field $v : \Omega \to \mathbb{R}^{d+1}$ solves the incompressible Euler equation subject to the acceleration of gravity, $-ge_y$ $(g > 0)$, while the free surface Σ is displaced by fluid particles. The vanishing of the normal velocity on Γ is simply the usual "solid-wall" condition and the pressure P vanishes on the free surface because we assume that there is no surface tension. Finally, the fluid is assumed to be incompressible and irrotational.

Notice that the pressure can be recovered from the velocity via the equation and by taking the divergence of (1.3), it satisfies

$$\Delta_{x,y} P = -\nabla_{x,y}^2 \cdot (v \otimes v).$$

As the motion of the liquid is supposed to be irrotational, the velocity field is therefore given by $v = \nabla_{x,y}\phi$ for some velocity potential satisfying

$$\Delta_{x,y}\phi = 0 \quad \text{in } \Omega, \qquad \partial_n \phi = 0 \quad \text{on } \Gamma.$$

Using the Bernoulli integral of the dynamical equations to express the pressure, the condition $P = 0$ on the free surface implies that

$$\begin{cases} \partial_t \eta = \partial_y \phi - \nabla\eta \cdot \nabla\phi & \text{on } \Sigma, \\ \partial_t \phi + \frac{1}{2} |\nabla_{x,y}\phi|^2 + gy = 0 & \text{on } \Sigma, \\ \partial_n \phi = 0 & \text{on } \Gamma, \end{cases} \tag{1.4}$$

where recall that $\nabla = \nabla_x$.

Introduce the so-called Taylor coefficient

$$a(t,x) = -(\partial_y P)(t,x,\eta(t,x)).$$

The stability of the waves is dictated by the Taylor sign condition, which is the assumption that there exists a positive constant c such that

$$a(t,x) \geq c > 0. \tag{1.5}$$

This assumption is now classical and we refer to [13, 24, 42, 67, 68] for various comments about this assumption. Here we only recall some basic facts. First of all, it is known that this assumption is propagated by the equation, so that this is an assumption about the initial data. Secondly, as proved by Wu, this assumption is automatically satisfied in the infinite depth case (that is when $\Gamma = \emptyset$) or for flat bottoms (when $\Gamma = \{y = -k\}$). There are two other cases where this assumption is known to be satisfied. For instance under a smallness assumption. Indeed, if $\partial_t \phi = O(\varepsilon^2)$ and $\nabla_{x,y}\phi = O(\varepsilon)$ then directly from the definition of the pressure we have $P + gy = O(\varepsilon^2)$. Secondly, it was proved by Lannes (see [42]) that the Taylor's assumption is satisfied under a smallness assumption on the curvature of the bottom (provided that the bottom is at least C^2). However, for general bottom we have to make an assumption on the Taylor coefficient.

1.1. The Zakharov/Craig-Sulem-Sulem system

Following Craig, Sulem and Sulem [28], we reduce the analysis to a system on the free surface $\Sigma(t) = \{y = \eta(t,x)\}$.

If ψ is defined by

$$\psi(t,x) = \phi(t,x,\eta(t,x)),$$

then ϕ is the unique variational solution of

$$\Delta\phi = 0 \text{ in } \Omega, \quad \phi|_\Sigma = \psi, \quad \partial_n \phi = 0 \text{ on } \Gamma.$$

Define the Dirichlet-Neumann operator by

$$\begin{aligned}(G(\eta)\psi)(t,x) &= \sqrt{1+|\nabla\eta|^2}\,\partial_n\phi|_{y=\eta(t,x)} \\ &= (\partial_y\phi)(t,x,\eta(t,x)) - \nabla\eta(t,x)\cdot(\nabla\phi)(t,x,\eta(t,x)).\end{aligned}$$

Now (η,ψ) solves (see [41, chapter 1])

$$\begin{cases} \partial_t\eta - G(\eta)\psi = 0, \\ \partial_t\psi + g\eta + \dfrac{1}{2}|\nabla\psi|^2 - \dfrac{1}{2}\dfrac{\big(\nabla\eta\cdot\nabla\psi + G(\eta)\psi\big)^2}{1+|\nabla\eta|^2} = 0. \end{cases} \tag{1.6}$$

1.2. Various unknowns

We shall work below with the horizontal and vertical traces of the velocity on the free boundary, namely

$$B = (\partial_y\phi)|_{y=\eta}, \quad V = (\nabla_x\phi)|_{y=\eta}.$$

These can be defined only in terms of η and ψ by means of the formula

$$B := \frac{\nabla\eta\cdot\nabla\psi + G(\eta)\psi}{1+|\nabla\eta|^2}, \qquad V := \nabla\psi - B\nabla\eta. \tag{1.7}$$

In turn, $\nabla_x\psi$ is expressed in terms of (V, B):

$$\nabla_x\psi = V + B\nabla_x\eta$$

and consequently (1.6) can be viewed as an equation on (V, B).

1.3. Known results

Our goal is to prove the existence of classical solutions (η, v) defined on some time interval $[0, T]$ such that there exist positive constants c and h such that conditions (1.2) and (1.5) hold for $0 \leq t \leq T$, assuming that these two conditions hold initially for $t = 0$. In terms of regularity threshold, our results are optimal, as long as dispersive effects are not taken into account.

Many results have been obtained on the Cauchy theory for the water-waves system, starting from the pioneering works of Nalimov [51], Yoshihara [70], Craig [29] (see also Hou, Teng and Zhang [36] and Beale, Hou and Lowengrub [14]). In the framework of Sobolev spaces and without smallness assumptions on the data, the well-posedness of the Cauchy problem was first proved by Beyer-Günther in [15] in the case with surface tension (in any number of space dimensions) and by Wu for the case without surface tension (see [67] for 2D water waves and [68] for the general case $d \geq 1$). Several extensions of their results have been obtained by different methods. We refer to the survey by Bardos and Lannes [13], the book by Lannes [41] and the survey paper of Craig and Wayne [31] for references and a short historical survey of the background of these problems. Here, we only review results about gravity water waves. For gravity-capillary waves, we refer the interested reader to Ambrose-Masmoudi [8, 10, 9], Schneider-Wayne [54], Schweizer [55], Iguchi [38, 37], Shatah-Zeng [57, 58], Coutand-Shkoller [27], Rousset-Tzvetkov [52], Christianson-Hur-Staffilani [23] and also our previous papers [2, 3]. In [24], Christodoulou–Lindblad proved *a priori* bounds in the case of nonvanishing vorticity in any number of space dimensions. The estimates are given in terms of geometric quantities. Assuming that an assumption similar to (1.5) is satisfied, they have shown that the Sobolev norms remain bounded essentially as long as the second fundamental form of the free surface is bounded, and the first-order derivatives of the velocity are bounded. As pointed out in the review paper of Craig and Wayne [31], such geometric estimates will be essential in the future progress of the study of singularity formation[1] in the fluid dynamics of free surface evolution. Based on the *a priori* estimates proved in [24], the existence of C^∞ solutions is proved in [46] together with an extension to the compressible case in [45].

In view of Cauchy-Lipschitz theorem, the Lipschitz regularity threshold for the velocity appears to be the natural assumption (as soon as no dispersion effects are taken into account). Indeed, this is necessary for the "fluid particles" to be well-defined. Our strategy is based on a direct analysis in Eulerian coordinates.

[1] At present, the only result about singularity formation is the recent breakthrough of Castro–Córdoba–Fefferman–Gancedo–Gómez Serrano [26] who exhibited smooth initial data for the 2D water wave equation for which smoothness of the interface breaks down in finite time (see also [25]).

In this direction it is influenced by the important paper by Lannes [42]. To some extent, our approach also contains the idea of using good vector fields. Thanks to the introduction of a new simple formulation of the equations, this reduces to proving good commutators estimated between a paraproduct and the convective derivative $\partial_t + V \cdot \nabla$.

Here we shall use tools from singular integrals analysis. In this direction, we follow the approach initiated by Craig–Schanz–Sulem [30] and further developped by Lannes [42]and Iooss–Plotnikov [39]. More precisely, we use paradifferential calculus, following Alazard-Métivier [1]. This approach is closely related to papers of Alinhac [4, 6] on rarefaction waves for hyperbolic systems. In particular, a key point in our analysis is to work with the so-called good unknown of Alinhac (see [1, 65]).

In the Half space, the Dirichlet Neumann operator takes the form

$$G(\eta)\Psi = |D_x|\Psi,$$

and it is easy to see that linearizing the system (1.6) at $\eta = 0, \Psi = 0$, we obtain

$$\partial_t \eta = |D_x|\Psi, \quad \partial_t \Psi = -g\eta,$$

and consequently the function

$$U = \eta + ig^{-1/2}|D_x|^{1/2}\Psi$$

satisfies the equation

$$\partial_t U = -ig^{1/2}|D_x|^{1/2}U. \tag{1.8}$$

In view of the dispersive properties of the linearized system (1.8), one can infer that solutions to whole water-waves system should enjoy similar properties. In [23], Christianson, Hur, and Staffilani initiated the study of the dispersive properties of the solutions of the water-wave system with surface tension and proved Strichartz-type estimates, for smooth-enough initial data. In [3], we prove such semi-classical Strichartz estimates (i.e. on time intervals tailored to the frequency), at the same low level of regularity we were able to construct the solutions in [2]. The proof of such estimates for dispersive equations with rough coefficients goes back to works of Smith [59], Tataru [61, 62, 63], Bahouri–Chemin [11], Staffilani–Tataru [60], and are also related to Burq–Gérard–Tvzetkov [20].

On the other hand, for smoother initial data, we proved that the solutions enjoy the optimal Strichartz estimates (i.e, without loss of regularity compared to the system linearized at the origin). Here our purpose is to go a little further and to take benefit of these Strichartz-type estimates to improve the regularity thresholds for the Cauchy theory. Notice finally that dispersive properties of the operator linearized at the origin together with normal form transforms, were used recently by Wu [69, 66] and Germain-Masmoudi-Shatah [32] to prove global existence results for for small localized waves.

2. Main results

2.1. Classical Cauchy theory

As explained above, a natural question is to prove well posedness under assumptions (in Sobolev spaces) which only ensure that the initial velocity has Lipschitz regularity (notice that in this problem, the notion of regularity has to be understood *up to the free surface*, as of course, the velocities field is analytic in the interior of the domain). This is the purpose of our first result, which reads as follows (let us recall that v is the velocities field in the interior of the domain, while V and B are the travces of the horizontal and vertical components of this velocities field).

Theorem 1. *Let $d \geq 1$, $s > 1 + d/2$ and consider an initial data (η_0, v_0) such that*

1. $\eta_0 \in H^{s+\frac{1}{2}}(\mathbf{R}^d), \quad V_0 \in H^s(\mathbf{R}^d), \quad B_0 \in H^s(\mathbf{R}^d),$

2. *there exists $h > 0$ such that condition* (1.2) *holds initially for $t = 0$,*

3. *there exists a positive constant c such that, for all $x \in \mathbf{R}^d$, $a_0(x) \geq c$.*

Then there exists $T > 0$ such that the Cauchy problem for (1.6) *with initial data (η_0, ψ_0) has a unique solution such that*

$$(\eta, V, B) \in C^0\Big([0, T]; H^{s+\frac{1}{2}}(\mathbf{R}^d) \times H^s(\mathbf{R}^d) \times H^s(\mathbf{R}^d)\Big)$$

such that

1. *the condition* (1.2) *holds for $0 \leq t \leq T$, with h replaced with $h/2$,*

2. *for all $0 \leq t \leq T$ and for all $x \in \mathbf{R}^d$, $a(t, x) \geq c/2$.*

2.2. Strichartz result

We can in fact take benefit of the dispersive properties of the water-waves system and improve the regularity thresholds just exhibited. Here for the sake of simplicity we only state our result in dimension $d = 2$ (recall that d is the dimension of the interface) without bottom (infinite depth) and we restrict our attention to a priori estimates (i.e. are not interested here in uniqueness issues).

Theorem 2. *Let $d = 2$, $s > 1 + d/2 - \frac{1}{12}$ and consider an initial data (η_0, v_0) such that*

$$\eta_0 \in H^{s+\frac{1}{2}}(\mathbf{R}^d), \quad V_0 \in H^s(\mathbf{R}^d), \quad B_0 \in H^s(\mathbf{R}^d).$$

Then there exists $T > 0$ such that the Cauchy problem for (1.6) *with initial data (η_0, v_0) has a solution (η, v) such that*

$$(\eta, V, B) \in C^0\Big([0, T]; H^{s+\frac{1}{2}}(\mathbf{R}^d) \times H^s(\mathbf{R}^d) \times H^s(\mathbf{R}^d)\Big)$$

(notice that in infinite depth, the Taylor sign condition is always satisfied).

2.3. Three-dimensional waves in a non-rectangular canal

We give here an illustration of the analysis of low regularity solutions in a domain with a rough boundary. We claim that the above analysis allows to prove the existence of 3D water gravity waves in a canal with vertical walls near the free surface. The propagation of waves whose crests are orthogonal to the walls is one of the main motivation for the analysis of 2D waves. It was historically at the heart of the analysis of water waves. The study of the propagation of three-dimensional water waves for the linearized equations goes back to Boussinesq (see [19]). However, there are no study of the the general case where the waves can be reflected on the walls of the canals, except the analysis of $3D$-periodic travelling waves which correspond to the reflexion of a 2D-wave off a vertical wall (see Reeder-Shinbrot [53] and Iooss-Plotnikov [39]).

More precisely, we consider a fluid domain which at time t is of the form

$$\Omega(t) = \{(x_1, x_2, y) \in (0,1) \times \mathbf{R} \times \mathbf{R} \,:\, b(x) < y < \eta(t,x)\},$$

for some given function b. We do not make any regularity assumption on b; again, our only assumption is that there exists a positive constant h such that $\eta(t,x) \geq b(x)+h$. We denote by Σ the free surface and by Γ the fixed boundary of the canal:

$$\Sigma(t) = \{(x_1, x_2, y) \in (0,1) \times \mathbf{R} \times \mathbf{R} \,:\, y = \eta(t,x)\},$$

and we set $\Gamma = \partial\Omega(t) \setminus \Sigma(t)$ (which does not depend on time).

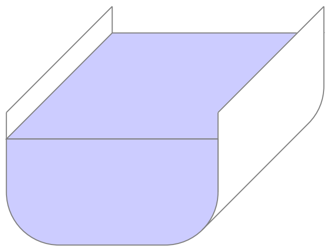

Figure 2.1: A non rectangular canal with vertical walls near the free surface.

Denote by n the normal to the boundary Γ and denote by ν the boundary to the free surface Σ. We begin with the following (elementary but seemingly new) observation: in the case of vertical walls, as long as the Taylor sign condition is satisfied, for the system (1.3) to be well posed, it is necessary that at the points where free surface and the boundary of the canal meet, the scalar product between the two normals (to the free surface and to the boundary of the canal) vanishes : $\nu \cdot n = 0$ on $\Sigma \cap \Gamma$, which means that the free surface Σ necessarily makes a right-angle with the rigid walls (see Figure 2.2).

Proposition 2.1. *Let (η, ϕ) be a solution of System (1.4) such that the Taylor coefficient a is continuous and positive. Then the angle between the free surface, $\Sigma(t)$ and the boundary of the canal Γ is a right angle:*

$$\forall t \in [0,T], \forall x \in \Sigma(t) \cap \Gamma, \quad n \cdot \nu(x) = 0.$$

Proof. Since $\nabla_{x,y} n = 0$ near the free surface, the boundary condition $\partial_n \phi = v \cdot n = 0$ implies that $[\partial_t v + (v \cdot \nabla_{x,y})v] \cdot n = 0$ near the free surface. It follows from the Euler equation that

$$\nabla_{x,y} P \cdot n = 0 \quad \text{near } \Gamma.$$

On the other hand, by assumption, the pressure is constant on the free surface and hence $\nabla_{x,y} P$ is proportional to the normal to Σ, ν. Notice now that Taylor sign condition reads $\partial_y P \mid_\Sigma < 0$ and concequently $\nabla_{x,y} P \mid_\Sigma \neq 0$. This implies that $\nu \cdot n = 0$ on $\Sigma \cap \Gamma$. □

Remark 2.2. Notice that the calculation above uses crucially that on the free surface the pressure P is constant. It does not apply in the presence of surface tension. Notice also that we used that along the flow, the time derivative of the normal to the canal vanishes: this result is also specific to the case of a flat vertical wall: it does not apply in the case of a curved wall.

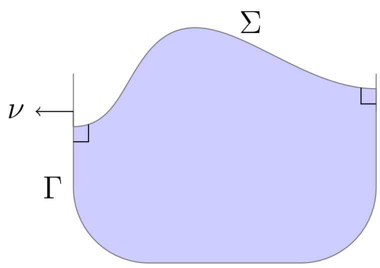

Figure 2.2: Two-dimensional section of the fluid domain, exhibiting the right-angles at the interface $\Sigma \cap \Gamma$

This suggests, following Boussinesq (see [19, page 37]) to perform a symmetrization process. Denote by $(\underline{\eta}, \underline{V})$ the functions thus obtained by symmetry and periodization (following the process which is illustrated on Figure 2.3). Of course, the symmetry would yield in general a Lipschitz singularity. However, that the possible singularities are weaker since

- the above physical observation about the right angles at the interface implies that $\partial_{x_1}\eta(t,0,x_2) = 0$,
- the solid wall boundary condition means that similarly $\partial_{x_1}\psi(t,0,x_2) = 0$.

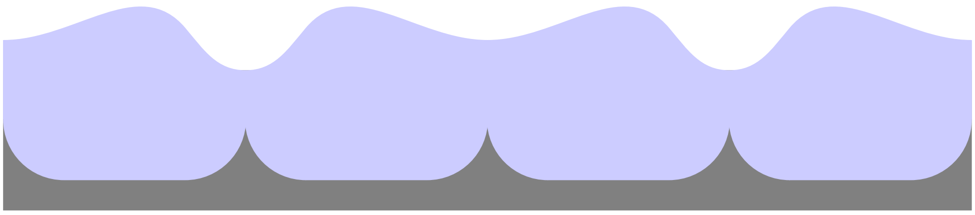

Figure 2.3: Two-dimensional section of the fluid domain after symmetry and periodization.

Theorem 3. *Let $\sigma \in (3, 7/2)$ and consider two functions $\eta_0, \psi_0 \colon (0,1) \times \mathbf{R} \to \mathbf{R}$ such that*

1. $\eta_0 \in H^\sigma((0,1)\times\mathbf{R}), \quad \psi_0 \in H^\sigma((0,1)\times\mathbf{R})$,
2. $\partial_{x_1}\eta_0(0,x_2) = \partial_{x_1}\eta_0(1,x_2) = 0$ *for all* $x_2 \in \mathbf{R}$,
3. $\partial_{x_1}\psi_0(0,x_2) = \partial_{x_1}\psi_0(1,x_2) = 0$ *for all* $x_2 \in \mathbf{R}$.

Then there exists a time $T > 0$ and two functions $\eta, \psi \colon [0,T] \times \mathbf{R}^2 \to \mathbf{R}$ such that

(a) (η, ψ) is a classical solution of (1.6);

(b) $\eta(t,x_1,x_2)$ and $\psi(t,x_1,x_2)$ are 2-periodic in x_1;

(c) with $s = \sigma - 1$ *we have*

$$\eta \in C^0([0,T]; H^{s+\frac{1}{2}}(\mathbf{T}\times\mathbf{R}), \quad \psi \in C^0([0,T]; H^s(\mathbf{T}\times\mathbf{R})),$$

(d) for all $(t, x_2) \in [0,T] \times \mathbf{R}$*, we have*

$$\partial_{x_1}\psi(t,0,x_2) = \partial_{x_1}\psi(t,1,x_2) = 0;$$

(e) for all $(x_1, x_2) \in \mathbf{R}^2$*, we have*

$$\eta(0,x_1,x_2) = \eta_0(x_1,x_2), \quad \psi(0,x_1,x_2) = \psi_0(x_1,x_2).$$

Proof. The proof is elementary : it consists in extending the initial data η_0, ψ_0, which are defined on $(0,1)\times\mathbf{R}$ to functions $(\underline{\eta}_0, \underline{\psi}_0)$ defined on the whole space $\mathbf{R}^2$. To do so, we proceed by a symmetry and a periodization. Then we apply our Cauchy theorem to these extended initial data. The key point is that, since we are able to prove the uniqueness for low regularity solutions, we can prove that the symmetry is propagated, which implies that the solutions satisfy the same symmetry as the initial data do. This implies that η, ψ are even in x_1 which implies that conclusion (d) holds.

There are various remarks that have to be made to see that one can apply the previous Cauchy theory.

1. First and foremost, we have to check that the symmetry/periodization process preserves the Sobolev norms. Of course, the symmetry would yield in general a Lipschitz singularity. However, the above physical observation about the right angles at the interface implies that $\partial_{x_1}\eta_0(0,x_2) = 0$, and hence that the possible singularities are weaker. More precisely, given any function $\eta_0 \in H^\mu((0,1)\times\mathbf{R})$, denote by $\underline{\eta}_0$ the periodic function in x_1, with period 2, such that

   $$\begin{aligned} \underline{\eta}_0(x_1,x_2) &= \eta_0(x_1,x_2) && \text{for } x_1 \in [0,1), \\ \underline{\eta}_0(x_1,x_2) &= \eta_0(-x_1,x_2) && \text{for } x_1 \in (-1,0]. \end{aligned}$$

   The condition $\partial_{x_1}\eta_0(0,x_2) = 0$ implies that

   $$\eta_0 \in H^\mu((0,1)_{x_1}\times\mathbb{R}_{x_2}) \Rightarrow \underline{\eta}_0 \in H^\mu((-1,1)_{x_1}\times\mathbb{R}_{x_2})$$

   as long as $\mu < 7/2$. Here the assumptions coming from Theorem 1 are $\eta_0 \in H^{s+\frac{1}{2}}, s > 2$. So that for $2 < s < 3$, we have $\underline{\eta}_0 \in H^{s+\frac{1}{2}}(\mathbf{T}\times\mathbf{R})$. A similar argument can be done for the velocities field.

2. To handle non rectangular canal, we have to work with rough bottoms. Here, even though initially the bottom is smooth, after symmetry/periodization this is no more the case, as Lipshitz or even cusps singularities can occur.

3. We cannot immediately apply Theorem 1, as of course, the only function in the space $H^s(\mathbf{R}^2_{x_1,x_2})$ which is periodic in the x_1 variable is the null function. However, an inspection of the proof of Theorem 1 shows that the result applies when the spatial variable x belongs to $\mathbf{T}\times\mathbf{R}$ instead of $\mathbf{R}^2$. Then we obtain the existence of a classical solution $(\underline{\eta}, \underline{\psi})$ to the water-wave system such that, initially $\underline{\eta}|_{t=0, x\in(0,1)\times\mathbf{R}} = \eta_0$ and $V|_{t=0} = V_0$.

4. One has to check that the symmetries are propagated, which ensures that the "solid wall" condition is satisfied. By the uniqueness part of the theorem, and the fact that the equations are invariant under the symmetrization process, we see that in $(-1,1)_{x_1} \times \mathbf{R}_{x_2}$, since the initial data η_0, ψ_0 are even in the x_1 variable, the solution (η, ψ) satisfies the same symmetry property. In particular, we easily check that, by restricting this solution to $(0,1) \times \mathbf{R}$, we obtain a solution of the water wave equation in the above canal (i.e. the condition $v \cdot n \mid_\Gamma = 0$ holds).

□

2.4. Swimming pools

The same analysis can be performed if instead of considering a canal (i.e. a free surface which is a graph over an infinite band $(0,1) \times \mathbf{R}$, on considers a swimming pool (i.e. a free surface which is a graph over $(0,1) \times (a,b)$.

3. Paradifferential calculus: a brief overview

Let us review notations and results about Bony's paradifferential calculus. We refer to [18, 35, 48, 50, 64] for the general theory. Here we follow the presentation by Métivier in [48] (which gives sharp operator norm estimates in terms of the seminorms of the symbols).

3.1. Paradifferential operators

For $k \in \mathbf{N}$ we denote by $W^{k,\infty}(\mathbf{R}^d)$ the usual Sobolev spaces. For $\rho \in]0,1[$ we denote by $W^{\rho,\infty}(\mathbf{R}^d)$ the space of bounded functions which are uniformly Hölder continuous with exponent ρ.

Definition 3.1. *Given $\rho \in [0,1]$ and $m \in \mathbf{R}$, $\Gamma^m_\rho(\mathbf{R}^d)$ denotes the space of locally bounded functions $a(x,\xi)$ on $\mathbf{R}^d \times (\mathbf{R}^d \setminus 0)$, which are C^∞ with respect to ξ for $\xi \neq 0$ and such that, for all $\alpha \in \mathbf{N}^d$ and all $\xi \neq 0$, the function $x \mapsto \partial_\xi^\alpha a(x,\xi)$ belongs to $W^{\rho,\infty}(\mathbf{R}^d)$ and there exists a constant C_α such that,*

$$\forall \, |\xi| \geq \frac{1}{2}, \quad \left\| \partial_\xi^\alpha a(\cdot,\xi) \right\|_{W^{\rho,\infty}} \leq C_\alpha (1+|\xi|)^{m-|\alpha|}. \tag{3.1}$$

Then $\dot{\Gamma}^m_\rho(\mathbf{R}^d)$ denotes the subspace of $\Gamma^m_\rho(\mathbf{R}^d)$ which consists of symbols $a(x,\xi)$ which are homogeneous of degree m with respect to ξ.

Given a symbol a, we define the paradifferential operator T_a by

$$\widehat{T_a u}(\xi) = (2\pi)^{-d} \int \chi(\xi-\eta,\eta) \widehat{a}(\xi-\eta,\eta) \psi(\eta) \widehat{u}(\eta) \, d\eta, \tag{3.2}$$

where $\widehat{a}(\theta,\xi) = \int e^{-ix\cdot\theta} a(x,\xi) \, dx$ is the Fourier transform of a with respect to the first variable; χ and ψ are two fixed C^∞ functions such that:

$$\psi(\eta) = 0 \quad \text{for } |\eta| \leq 1, \qquad \psi(\eta) = 1 \quad \text{for } |\eta| \geq 2, \tag{3.3}$$

and $\chi(\theta,\eta)$ is homogeneous of degree 0 and satisfies, for $0 < \varepsilon_1 < \varepsilon_2$ small enough,

$$\chi(\theta,\eta) = 1 \quad \text{if} \quad |\theta| \leq \varepsilon_1 |\eta|, \qquad \chi(\theta,\eta) = 0 \quad \text{if} \quad |\theta| \geq \varepsilon_2 |\eta|.$$

3.2. Symbolic calculus

We shall use quantitative results from [48] about operator norms estimates in symbolic calculus. To do so, introduce the following semi-norms.

Definition 3.2. *For $m \in \mathbf{R}$, $\rho \in [0,1]$ and $a \in \Gamma^m_\rho(\mathbf{R}^d)$, we set*

$$M^m_\rho(a) = \sup_{|\alpha| \le \frac{d}{2}+1+\rho} \sup_{|\xi| \ge 1/2} \left\| (1+|\xi|)^{|\alpha|-m} \partial^\alpha_\xi a(\cdot,\xi) \right\|_{W^{\rho,\infty}(\mathbf{R}^d)}. \tag{3.4}$$

The main features of symbolic calculus for paradifferential operators are given by the following theorem.

Definition 3.3. *Let $m \in \mathbf{R}$. An operator T is said of order m if, for all $\mu \in \mathbf{R}$, it is bounded from H^μ to $H^{\mu-m}$.*

Theorem 4. *Let $m \in \mathbf{R}$ and $\rho \in [0,1]$.*

(i) If $a \in \Gamma^m_0(\mathbf{R}^d)$, then T_a is of order m. Moreover, for all $\mu \in \mathbf{R}$ there exists a constant K such that

$$\|T_a\|_{H^\mu \to H^{\mu-m}} \le K M^m_0(a). \tag{3.5}$$

(ii) If $a \in \Gamma^m_\rho(\mathbf{R}^d), b \in \Gamma^{m'}_\rho(\mathbf{R}^d)$ then $T_aT_b - T_{ab}$ is of order $m+m'-\rho$. Moreover, for all $\mu \in \mathbf{R}$ there exists a constant K such that

$$\|T_aT_b - T_{ab}\|_{H^\mu \to H^{\mu-m-m'+\rho}} \le K M^m_\rho(a) M^{m'}_0(b) + K M^m_0(a) M^{m'}_\rho(b). \tag{3.6}$$

(iii) Let $a \in \Gamma^m_\rho(\mathbf{R}^d)$. Denote by $(T_a)^*$ the adjoint operator of T_a and by $\overline{a}$ the complex-conjugated of a. Then $(T_a)^* - T_{\overline{a}}$ is of order $m-\rho$. Moreover, for all μ there exists a constant K such that

$$\|(T_a)^* - T_{\overline{a}}\|_{H^\mu \to H^{\mu-m+\rho}} \le K M^m_\rho(a). \tag{3.7}$$

Remark 3.4. With regards to symbolic composition (see the second point in the theorem above), a difference with our previous work [2] is that we do not need here to take into account sub-principal symbols. Therefore it is enough for our purposes to consider the simplest case where symbolic composition reduces to the product of symbols. Another difference is that we shall need sharp quantitative bounds which are linear with respect to $M^m_\rho(a)$ and $M^m_\rho(b)$ instead of being quadratic (for the proof of (3.6), we refer the reader to [48, Thm 6.1.4]).

Remark 3.5. We also have analogous results in Hölder spaces. For instance, for all $m, \mu \ge 0$ such that $\mu \notin \mathbf{N}$ and $\mu + m \notin \mathbf{N}$, there exists a constant K such that

$$\|T_a\|_{W^{\mu+m,\infty} \to W^{\mu,\infty}} \le K M^0_0(a). \tag{3.8}$$

3.3. Paraproducts and product rules

If $a = a(x)$ is a function of x only, the paradifferential operator T_a is called a paraproduct. A key feature of paraproducts is that one can replace nonlinear expressions by paradifferential expressions, to the price of error terms which are smoother than the main terms. Also, one can define paraproducts T_a for rough functions a which do not belong to L^∞ but merely $H^{d/2-m}(\mathbf{R}^d)$ with $m > 0$.

Definition 3.6. *Given two functions a, b defined on $\mathbf{R}^d$ we define the remainder*

$$R(a,u) = au - T_a u - T_u a.$$

Definition 3.7. *Consider a dyadic decomposition of the identity:* $I = \Delta_{-1} + \sum_{q=0}^{\infty} \Delta_q$. *If* s *is any real number, we define the Zygmund class* $C_*^s(\mathbf{R}^d)$ *as the space of tempered distributions* u *such that*

$$\|u\|_{C_*^s} := \sup_q 2^{qs} \|\Delta_q u\|_{L^\infty} < +\infty.$$

Remark 3.8. It is known that $C_*^s = W^{s,\infty}$ if $s > 0$ is not an integer. Below, to simplify notations, we drop the star and simply denote by C^s the Zygmund spaces.

We record here various estimates about paraproducts (see chapter 2 in [12] or [22]).

Theorem 5. (*i*) *Let* $\alpha, \beta \in \mathbf{R}$. *If* $\alpha + \beta > 0$ *then*

$$\|R(a,u)\|_{H^{\alpha+\beta-\frac{d}{2}}(\mathbf{R}^d)} \leq K \|a\|_{H^\alpha(\mathbf{R}^d)} \|u\|_{H^\beta(\mathbf{R}^d)}, \tag{3.9}$$

$$\|R(a,u)\|_{C^{\alpha+\beta}(\mathbf{R}^d)} \leq K \|a\|_{C^\alpha(\mathbf{R}^d)} \|u\|_{C^\beta(\mathbf{R}^d)}, \tag{3.10}$$

$$\|R(a,u)\|_{H^{\alpha+\beta}(\mathbf{R}^d)} \leq K \|a\|_{C^\alpha(\mathbf{R}^d)} \|u\|_{H^\beta(\mathbf{R}^d)}. \tag{3.11}$$

(*ii*) *Let* $m > 0$ *and* $s \in \mathbf{R}$. *Then*

$$\|T_a u\|_{H^{s-m}} \leq K \|a\|_{C^{-m}} \|u\|_{H^s}, \tag{3.12}$$

$$\|T_a u\|_{C^{s-m}} \leq K \|a\|_{C^{-m}} \|u\|_{C^s}. \tag{3.13}$$

(*iii*) *Let* s_0, s_1, s_2 *be such that* $s_0 \leq s_2$ *and* $s_0 < s_1 + s_2 - \frac{d}{2}$, *then*

$$\|T_a u\|_{H^{s_0}} \leq K \|a\|_{H^{s_1}} \|u\|_{H^{s_2}}. \tag{3.14}$$

4. The main steps in the proof

4.1. The main difficulties

Clearly, the main difficulty in our analysis is that we consider low regularity solutions. As we shall see, this raises new interesting questions which would be easily solved by considering $s > 3/2 + d/2$. Let us give three examples of problems which we need to solve.

- Some problems come from the fact that many coefficients belong to the Sobolev spaces $H^{s-1/2}(\mathbf{R}^d)$. For $s > 3/2 + d/2$, these terms belong to the space of C^1 functions. As a result, the commutator between these terms and a differential operator of order m can be handled as an operator of order $m-1$. However, for $s > 1 + d/2$ these terms only belong to $C^{1/2}$ and hence the commutators can only be handled as operators of order $m - 1/2$.

- If $V \in H^s$ then $\nabla V \in H^{s-1}$. However, H^{s-1} is not an algebra.

- We use in an essential way the fact that $\partial_t + V \cdot \nabla$ has the same weight as $|D_x|^{1/2}$. Indeed, we have many coefficients which belong only to $C^{1/2}$. Then the derivatives with respect to time, or with respect to the vector field $V \cdot \nabla$ only belong to $C^{-1/2}$, which lead to loss of derivative. Instead we shall see that the convective derivative of these coefficients belongs to L^∞.

- In view of the Strichartz result (Theorem 2, we cannot assume that $s > 1+d/2$. Instead we only assume that $s > a + d/2$ for some $a < 1$ and we assume an a priori estimate of the Lipschitz norm (which comes from Strichartz estimates).

4.2. The strategy

4.2.1. Paralinearisation of the Dirichlet Neumann operator

Introduce

$$U = V + T_{\nabla_x \eta} B.$$

A crucial step in the proof is the following

Proposition 4.1. *Assume that* (η, V, B) *are smooth solutions of the system* (1.6)*. Then There exists a non decreasing function* C *such that*

$$(\partial_t + T_V \cdot \nabla) U + T_a \zeta = f_1,$$
$$(\partial_t + T_V \cdot \nabla) \zeta = T_\lambda U + f_2,$$

where, for each time $t \in [0, T]$*,*

$$\begin{aligned}&\left\| (f_1(t), f_2(t)) \right\|_{H^s \times H^{s-\frac{1}{2}}} \\ &\qquad \leq C \left(\left\| \eta(t) \right\|_{H^{s+\frac{1}{2}}}, \left\| (V,B)(t) \right\|_{H^s} \right) \left\{ 1 + \left\| \eta(t) \right\|_{C^{\frac{3}{2}}} + \left\| (V,B)(t) \right\|_{C^r} \right\}.\end{aligned}$$

Remark 4.2. The new unknown U is related to what is called the good-unknown of Alinhac in [1, 2]. It originates in the works by Alinhac [4, 6]. The proof of this result requires a careful analysis of the Dirichlet-Neumann operator (see Section 4.3)

4.2.2. A priori estimates

To prove the well-posedness, we follow a standard approach. With regards to the existence, we obtain solutions to the system (1.6) as limits of smooth solutions to approximate systems. This approach has been detailed in [2]. One has to notice that this step is actually much easier without surface tension. One reason is that with surface tension, we require some mollifiers with various properties (good estimates for commutators with the principal part of the operator). Here it is possible to use the simplest ones (which are convolutions operators) since the reduced paradifferential system involves only operator of order less than equal to 1.

The key point is to prove that solutions of the approximate system are uniformly bounded with respect to ε. To do so, as in [2], it is enough to prove *a priori* estimates.

4.2.3. Convergence

Once it is granted that approximate solutions exist and are uniformly bounded on a time interval independent of ε, one has to prove that they converge to a solution of the original system. To do this, one cannot apply standard compactness results since the Dirichlet-Neumann operator is not a local operator. To overcome this difficulty we shall prove as in [2, 42] that

1. The solutions $(\eta_\varepsilon, \psi_\varepsilon)$ form a Cauchy sequence in an appropriate bigger space (by an estimate of the difference of two solutions $(\eta_\varepsilon, \psi_\varepsilon)$ and $(\eta_{\varepsilon'}, \psi_{\varepsilon'})$).

2. (η, ψ) is a solution to (1.6).

5. $(\eta, \psi) \in C^0([0, T]; H^{s+\frac{1}{2}}(\mathbf{R}^d) \times H^s(\mathbf{R}^d))$.

4.2.4. Strichartz estimates

The proof of Theorem 2 relies also on Proposition 4.1. We need however to combine additional reductions and a priori estimates with a bootstrap argument. The first step is classical in the context of quasi-linear wave equations (see the works by Lebeau [43], Smith [59], Bahouri-Chemin [11] and Tataru [61]). It consists, after a dyadic decomposition at frequency h^{-1}, in regularizing the coefficients at scale $h^{-\delta}, \delta \in (0,1)$. Then, we need to straighten the vector field $\partial_t + V\nabla_x$ by means of a para-change of variables (see [4]). Finally, we are able to write a parametrix for the reduced system, which allows to prove Strichartz estimates using the usual strategy (TT^* method) on a small time interval $|t| \leq h^{\delta}$. To conclude, it suffices to glue the estimates.

4.3. Paralinearization of the Dirichlet-Neumann operator

To simplify the reading, let us consider the case of infinite depth. It follows from a variational analysis that, given a function $f\colon \mathbf{R}^d \to \mathbf{R}$, one can define a unique variational solution ϕ of

$$\Delta_{x,y}\phi = 0 \quad \text{in } \Omega, \qquad \phi|_{y=\eta(x)} = f. \tag{4.1}$$

Then

$$\int_\Omega |\nabla_{x,y}\phi|^2 \, dxdy \leq K \left\Vert f \right\Vert^2_{H^{1/2}(\mathbf{R}^d)}, \tag{4.2}$$

for some constant K depending only on the Lipschitz norm of η. See [41] for the case without bottom. For the case with a general bottom, this follows from the analysis in [2].

Then we define the Dirichlet-Neumann operator, denoted by $G(\eta)$, by

$$G(\eta)f = \sqrt{1+|\nabla\eta|^2}\, \partial_n \phi\Big|_{y=\eta(x)} = \Big[\partial_y\phi - \nabla\eta\cdot\nabla\phi\Big]\Big|_{y=\eta(x)}.$$

It is known that this operator is well defined under general assumptions (see [41] or [30]). The fact that this operator is well-defined under general assumption on the bottom was proved in our previous paper [2]. Using the Fourier transform, it is easily seen that $G(0)$ is the Fourier multiplier $|D_x|$. More generally, if η is a smooth function, then it is known since Calderón that $G(\eta)$ is a pseudo-differential operator whose principal symbol is given by

$$\lambda(x,\xi) := \sqrt{(1+|\nabla\eta(x)|^2)\,|\xi|^2 - (\nabla\eta(x)\cdot\xi)^2}.$$

Notice that λ is well-defined for any C^1 function η. The main result of this section allows to compare $G(\eta)$ to the para-differential operator T_λ when η has limited regularity. Namely we want to estimate the operator

$$R(\eta) = G(\eta) - T_\lambda.$$

It follows from the analysis in [1, 2] that we have the following estimates.

Proposition 4.3. *If $s > 2 + d/2$ then*

$$\left\Vert R(\eta)f\right\Vert_{H^s} \leq C\left(\left\Vert \eta\right\Vert_{H^{s+1}}\right)\left\Vert f\right\Vert_{H^s}, \tag{4.3}$$

$$\left\Vert R(\eta)f\right\Vert_{H^{s-\frac12}} \leq C\left(\left\Vert \eta\right\Vert_{H^{s+\frac12}}\right)\left\Vert f\right\Vert_{H^s}. \tag{4.4}$$

Remark 4.4. The first bound is related to the fact that $R(\eta)$ is expected to be an operator of order 0, which means that $R(\eta)f$ and f are expected to have the same regularity, at least when η is much smoother than f. The main interest of the first bound is to show that $R(\eta)f$ and f have the same regularity precisely when f is only one derivative less regular than η. Now, if η is only one-half derivative smoother than f, as assumed in (4.4), then *the regularity of $R(\eta)f$ is dictated by the regularity of η instead of the regularity of f* (as proved by the formulas given in [1, 2]).

Here we need a sharp result, where one only assumes that $s > 3/4 + d/2$, instead of $s > 2 + d/2$, together with an a priori bound in Hölder spaces C^r ($r > 1$, $r \notin \mathbf{N}$). This is tailored to the Strichartz norm.

Theorem 6. *Let $d \geq 1$ and consider $s, r \in \mathbf{R}$ such that*

$$s > \frac{3}{4} + \frac{d}{2}, \qquad r > 1.$$

Consider $\eta \in H^{s+\frac{1}{2}}(\mathbf{R}^d) \cap C^{\frac{3}{2}}(\mathbf{R}^d)$ and $f \in H^s(\mathbf{R}^d) \cap C^r(\mathbf{R}^d)$, then

$$R(\eta)f \in H^{s-\frac{1}{2}}(\mathbf{R}^d).$$

Moreover

$$\|R(\eta)f\|_{H^{s-\frac{1}{2}}} \leq C\left(\|\eta\|_{H^{s+\frac{1}{2}}}, \|f\|_{H^s}\right)\left\{1 + \|\eta\|_{C^{\frac{3}{2}}} + \|f\|_{C^r}\right\}, \tag{4.5}$$

for some continuous function $C \colon (\mathbf{R}^+)^2 \to \mathbf{R}^+$ depending only on s and r.

Remark 4.5. The main interest of this result is that the right-hand side of (4.5) is linear with respect to the "highest" norms. To compare, say, the norms $\|\cdot\|_{H^{s+\frac{1}{2}}}$ and $\|\cdot\|_{C^{3/2}}$, the heuristic argument is that for $s < 1 + d/2$,

$$\left\|u\left(\frac{x}{\varepsilon}\right)\right\|_{C^{\frac{3}{2}}} \sim \left(\frac{1}{\varepsilon}\right)^{3/2} >> \left(\frac{1}{\varepsilon}\right)^{s+\frac{1}{2}-\frac{d}{2}} \sim \left\|u\left(\frac{x}{\varepsilon}\right)\right\|_{H^{s+\frac{1}{2}}}.$$

Another issue in the proof is to deal with rough data. For example it would have been much easier to prove that for $s \geq s_0$ with s_0 large enough we have the following tame estimate

$$\|R(\eta)f\|_{H^{s-\frac{1}{2}}} \leq C\left(\|\eta\|_{H^{s_0+\frac{1}{2}}}, \|f\|_{H^{s_0}}\right)\left\{1 + \|\eta\|_{H^{s+\frac{1}{2}}} + \|f\|_{H^s}\right\}. \tag{4.6}$$

Notice that (4.6) would be useless for the purpose of improving the Cauchy theory by means of Strichartz estimates.

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