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Abstract

We consider a second-order selfadjoint elliptic operator with an anisotropic diffusion matrix having a jump across a smooth hypersurface. We prove the existence of a weight-function such that a Carleman estimate holds true. We moreover prove that the conditions imposed on the weight function are necessary.

1. Introduction

1.1. Carleman estimates

Let $P(x, D_x)$ be a differential operator defined on some open subset of $\mathbb{R}^n$. A *Carleman estimate* for this operator is the following weighted a priori inequality

$$\|e^{\tau\varphi} P w\|_{L^2(\mathbb{R}^n)} \gtrsim \|e^{\tau\varphi} w\|_{L^2(\mathbb{R}^n)}, \tag{1.1}$$

where the weight function φ is real-valued with a non-vanishing gradient, τ is a large positive parameter and w is any smooth compactly supported function. This type of estimate was used for the first time in 1939 in T. Carleman's article [5] to handle uniqueness properties for the Cauchy problem for non-hyperbolic operators. To this day, it remains essentially the only method to prove unique continuation properties for ill-posed problems, in particular to handle uniqueness of the Cauchy problem for elliptic operators with non-analytic coefficients. This tool has been refined, polished and generalized by manifold authors and plays now a very important rôle in control theory and inverse problems. The 1958 article by A.P. Calderón [4] gave a very important development of the Carleman method with a proof of an estimate of the form of (1.1) using a pseudo-differential factorization of the operator, giving a new start to singular-integral methods in local analysis. In the article [7] and in

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his first PDE book (Chapter VIII, [8]), L. Hörmander showed that local methods could provide the same estimates, with weaker assumptions on the regularity of the coefficients of the operator.

For instance, for second-order elliptic operators with real coefficients[1] in the principal part, Lipschitz continuity of the coefficients suffices for a Carleman estimate to hold and thus for unique continuation across a $\mathscr{C}^1$ hypersurface. Naturally pseudo-differential methods require more derivatives, at least tangentially, i.e., essentially on each level surface of the weight function φ. Chapters 17 and 28 in the 1983-85 four-volume book [9] by L. Hörmander contain more references and results.

Furthermore, it was shown by A. Pliś [18] that Hölder continuity is not enough to get unique continuation: this author constructed a real homogeneous linear differential equation of second order and of elliptic type on $\mathbb{R}^3$ without the unique continuation property although the coefficients are Hölder-continuous with any exponent less than one. The construction by K. Miller in [17] showed that Hölder continuity[2] is not enough to obtain unique continuation for second-order elliptic operators, even in divergence form (see also the 1998 articles [3], [19] for the particular 2D case where boundedness is essentially enough to get unique continuation for elliptic equations in the case of $W^{1,2}$ solutions).

1.2. Jump discontinuities

Although the situation seems to be almost completely clarified by the previous results, with a minimal and somewhat necessary condition on Lipschitz continuity, we are interested in the following second-order elliptic operator $\mathcal{L}$,

$$\mathcal{L}w = -\operatorname{div}(A(x)\nabla w),\ A(x) = (a_{jk}(x))_{1\le j,k\le n} = A^T(x),\ \inf_{\|\xi\|_{\mathbb{R}^n}=1}\langle A(x)\xi, \xi\rangle > 0, \tag{1.2}$$

in which the matrix A has a jump discontinuity across a smooth hypersurface. However we shall impose some stringent –yet natural– restrictions on the domain of functions w, which will be required to satisfy some *transmission conditions*, detailed in the next sections. Roughly speaking, it means that w must belong to the domain of the operator, with continuity at the interface, so that ∇w remains bounded and continuity of the flux across the interface, so that $\operatorname{div}(A\nabla w)$ remains bounded, avoiding in particular the occurrence of a simple or multiple layer at the interface.

The article [6] by A. Doubova, A. Osses, and J.-P. Puel tackled that problem, in the isotropic case (the matrix A is scalar $c\,\mathrm{Id}$) with a monotonicity assumption: the observation takes place in the region where the diffusion coefficient c is the 'lowest'. (Note that the work of [6] concerns the case of a parabolic operator but an adaptation to an elliptic operator is straightforward.) In the one-dimensional case, the monotonicity assumption was relaxed for general piecewise $\mathscr{C}^1$ coefficients by A. Benabdallah, Y. Dermenjian and J. Le Rousseau [2], and for coefficients with bounded variations [11]. The case of an arbitrary dimension without any monotonicity condition in the elliptic case was solved by J. Le Rousseau and L. Robbiano in [11]: there the isotropic case is treated as well as a particular case of anisotropic

[1]The paper [1] by S. Alinhac shows nonunique continuation property for second-order elliptic operators with non-conjugate roots; of course, if the coefficients of the principal part are real, this is excluded.

[2]The counterexample of [17] is Hölder continuous with index 1/6.

medium. An extension of their approach to the case of parabolic operators can be found in [15].

The purpose of the present article is to show that a Carleman estimate can be proven for any operator of type (1.2) without an isotropy assumption: $A(x)$ is a symmetric positive-definite matrix with a jump discontinuity across a smooth hypersurface. We also provide conditions on the Carleman weight function that are rather simple to handle and we prove that they are sharp.

The approach we follow differs from that of [11] where the authors base their analysis on the usual Carleman method for certain microlocal regions and on Calderón projectors for others. The regions they introduce are determined by the ellipticity or non-ellipticity of the conjugated operator. Here, our approach is somewhat closer to A. Calderón's original work on unique continuation [4]: the conjugated operator is factored out in first-order (pseudo-differential) operators for which estimates are derived. Naturally, the quality of these estimates depends on their elliptic or non-elliptic nature; we thus recover microlocal regions that correspond to that of [11]. Note that such a factorization is also used in [10] to prove Carleman estimates at a boundary in the case of non-homogeneous boundary conditions.

1.3. Notation and statement of the main result

Let Ω be an open subset of $\mathbb{R}^n$ and Σ be a $\mathscr{C}^\infty$ oriented hypersurface of Ω: we have the partition

$$\Omega = \Omega_+ \cup \Sigma \cup \Omega_-, \quad \overline{\Omega_\pm} = \Omega_\pm \cup \Sigma, \quad \Omega_\pm \text{ open subsets of } \mathbb{R}^n, \tag{1.3}$$

and we introduce the following Heaviside-type functions

$$H_\pm = \mathbf{1}_{\Omega_\pm}. \tag{1.4}$$

We consider the elliptic second-order operator

$$\mathcal{L} = D \cdot AD = -\operatorname{div}(A(x)\nabla), \qquad (D = -i\nabla), \tag{1.5}$$

where $A(x)$ is a symmetric positive-definite $n \times n$ matrix, such that

$$A = H_- A_- + H_+ A_+, \quad A_\pm \in \mathscr{C}^\infty(\Omega). \tag{1.6}$$

We shall consider functions w of the following type:

$$w = H_- w_- + H_+ w_+, \quad w_\pm \in \mathscr{C}^\infty(\Omega). \tag{1.7}$$

We have $dw = H_- dw_- + H_+ dw_+ + (w_+ - w_-)\delta_\Sigma \nu$, where δ_Σ is the Euclidean hypersurface measure on Σ and ν is the unit conormal vector field to Σ pointing into Ω_+. To remove the singular term, we assume

$$w_+ = w_- \quad \text{at } \Sigma, \tag{1.8}$$

so that $Adw = H_- A_- dw_- + H_+ A_+ dw_+$ and

$$\operatorname{div}(Adw) = H_- \operatorname{div}(A_- dw_-) + H_+ \operatorname{div}(A_+ dw_+) + \langle A_+ dw_+ - A_- dw_-, \nu\rangle \delta_\Sigma.$$

Moreover, we shall assume that

$$\langle A_+ dw_+ - A_- dw_-, \nu\rangle = 0 \quad \text{at } \Sigma, \text{ i.e. } \langle dw_+, A_+\nu\rangle = \langle dw_-, A_-\nu\rangle, \tag{1.9}$$

so that

$$\operatorname{div}(Adw) = H_- \operatorname{div}(A_- dw_-) + H_+ \operatorname{div}(A_+ dw_+). \tag{1.10}$$

Conditions (1.8)-(1.9) will be called *transmission conditions* on the function w and we define the vector space

$$\mathcal{W} = \{H_- w_- + H_+ w_+\}_{w_\pm \in \mathscr{C}^\infty(\Omega) \text{ satisfying (1.8)-(1.9)}}. \tag{1.11}$$

Note that (1.8) is a continuity condition of w across Σ and (1.9) is concerned with the continuity of $\langle Adw, \nu\rangle$ across Σ, i.e. the continuity of the flux of the vector field Adw across Σ. A weight function "suitable for observation from Ω_+" is defined as a Lipschitz continuous function φ on Ω such that

$$\varphi = H_-\varphi_- + H_+\varphi_+, \quad \varphi_\pm \in \mathscr{C}^\infty(\Omega), \quad \varphi_+ = \varphi_-, \quad \langle d\varphi_\pm, X\rangle > 0 \quad \text{at } \Sigma, \tag{1.12}$$

for any positively transverse vector field X to Σ (i.e. $\langle \nu, X\rangle > 0$).

Theorem 1.1. *Let $\Omega, \Sigma, \mathcal{L}, \mathcal{W}$ be as in* (1.3)*,* (1.5) *and* (1.11)*. Then for any compact subset K of Ω, there exist a weight function φ satisfying* (1.12) *and positive constants C, τ_1 such that for all $\tau \geq \tau_1$ and all $w \in \mathcal{W}$ with* $\operatorname{supp} w \subset K$*,*

$$\begin{aligned} C\|e^{\tau\varphi}\mathcal{L}w\|_{L^2(\mathbb{R}^n)} \geq \tau^{\frac{3}{2}}\|e^{\tau\varphi}w\|_{L^2(\mathbb{R}^n)} + \tau^{\frac{1}{2}}\|e^{\tau\varphi}\nabla w\|_{L^2(\mathbb{R}^n)} + \tau^{\frac{3}{2}}|(e^{\tau\varphi}w)_{|\Sigma}|_{L^2(\Sigma)} \\ + \tau^{\frac{1}{2}}|(e^{\tau\varphi}\nabla w_+)_{|\Sigma}|_{L^2(\Sigma)} + \tau^{\frac{1}{2}}|(e^{\tau\varphi}\nabla w_-)_{|\Sigma}|_{L^2(\Sigma)}. \end{aligned} \tag{1.13}$$

Remark 1.2. It is important to notice that whenever a true discontinuity occurs for the vector field $A\nu$, then the space $\mathcal{W}$ does *not* contain $\mathscr{C}^\infty(\Omega)$: the inclusion $\mathscr{C}^\infty(\Omega) \subset \mathcal{W}$ implies from (1.9) that for all $w \in \mathscr{C}^\infty(\Omega)$, $\langle dw, A_+\nu - A_-\nu\rangle = 0$ at Σ so that $A_+\nu = A_-\nu$ at Σ, that is continuity for $A\nu$. The Carleman estimate which is proven in the present paper takes naturally into account these transmission conditions on the function w and it is important to keep in mind that the occurrence of a jump is excluding many smooth functions from the space $\mathcal{W}$. On the other hand, we have $\mathcal{W} \subset \operatorname{Lip}(\Omega)$.

1.4. Sketch of the proof

We provide in this subsection an outline of the main arguments used in our proof. To avoid technicalities, we somewhat simplify the geometric data and the weight function, keeping of course the anisotropy. We consider the operator

$$\mathcal{L}_0 = \sum_{1\leq j\leq n} D_j c_j D_j, \quad c_j(x) = H_+ c_j^+ + H_- c_j^-, \tag{1.14}$$

$$c_j^\pm \text{ positive constants}, \quad H_\pm = \mathbf{1}_{\{\pm x_n > 0\}}, \tag{1.15}$$

with $D_j = \frac{\partial}{i\partial x_j}$, and the vector space $\mathcal{W}_0$ of functions $H_+w_+ + H_-w_-$, $w_\pm \in \mathscr{C}_c^\infty(\mathbb{R}^n)$, such that

$$\text{at } x_n = 0,\ w_+ = w_-,\ c_n^+\partial_n w_+ = c_n^-\partial_n w_- \ \text{(transmission conditions across } x_n = 0). \tag{1.16}$$

As a result, for $w \in \mathcal{W}_0$, we have $D_n w = H_+ D_n w_+ + H_- D_n w_-$ and

$$\mathcal{L}_0 w = \sum_j (H_+ c_j^+ D_j^2 w_+ + H_- c_j^- D_j^2 w_-). \tag{1.17}$$

We also consider a weight function[3]

$$\varphi = \underbrace{(\alpha_+ x_n + \beta x_n^2/2)}_{\varphi_+} H_+ + \underbrace{(\alpha_- x_n + \beta x_n^2/2)}_{\varphi_-} H_-, \quad \alpha_\pm > 0, \quad \beta > 0, \tag{1.18}$$

[3]In the main text, we shall introduce some minimal requirements on weight function and suggest other possible choices for φ.

a positive parameter τ and the vector space $\mathcal{W}_\tau$ of functions $H_+v_+ + H_-v_-$, $v_\pm \in \mathscr{C}_c^\infty(\mathbb{R}^n)$, such that at $x_n = 0$

$$v_+ = v_-, \tag{1.19}$$

$$c_n^+(D_n v_+ + i\tau\alpha_+ v_+) = c_n^-(D_n v_- + i\tau\alpha_- v_-). \tag{1.20}$$

Observe that $w \in \mathcal{W}_0 \Leftrightarrow v = e^{\tau\varphi} w \in W_\tau$. We have

$$e^{\tau\varphi}\mathcal{L}_0 w = \underbrace{e^{\tau\varphi}\mathcal{L}_0 e^{-\tau\varphi}}_{\mathcal{L}_\tau}(e^{\tau\varphi}w)$$

so that proving a weighted a priori estimate $\|e^{\tau\varphi}\mathcal{L}_0 w\|_{L^2(\mathbb{R}^n)} \gtrsim \|e^{\tau\varphi}w\|_{L^2(\mathbb{R}^n)}$ for $w \in \mathcal{W}_0$ amounts to getting $\|\mathcal{L}_\tau v\|_{L^2(\mathbb{R}^n)} \gtrsim \|v\|_{L^2(\mathbb{R}^n)}$ for $v \in \mathcal{W}_\tau$.

Step 1: pseudo-differential factorization. Using Einstein convention on repeated indices $j \in \{1, \dots, n-1\}$, we have

$$\mathcal{L}_\tau = (D_n + i\tau\varphi')c_n(D_n + i\tau\varphi') + D_j c_j D_j$$

and for $v \in \mathcal{W}_\tau$, from (1.17), with $m_\pm = m_\pm(D') = (c_n^\pm)^{-\frac{1}{2}}(c_j^\pm D_j^2)^{\frac{1}{2}}$,

$$\mathcal{L}_\tau v = H_+ c_n^+\Big((D_n + i\tau\varphi'_+)^2 + m_+^2\Big)v_+ + H_- c_n^-\Big((D_n + i\tau\varphi'_-)^2 + m_-^2\Big)v_-$$

so that

$$\begin{aligned}\mathcal{L}_\tau v = H_+ c_n^+\Big(D_n + i(\overbrace{\tau\varphi'_+ + m_+}^{e_+})\Big)\Big(D_n + i(\overbrace{\tau\varphi'_+ - m_+}^{f_+})\Big)v_+ \\ + H_- c_n^-\Big(D_n + i(\underbrace{\tau\varphi'_- - m_-}_{f_-})\Big)\Big(D_n + i(\underbrace{\tau\varphi'_- + m_-}_{e_-})\Big)v_-.\end{aligned} \tag{1.21}$$

Note that $e_\pm$ are elliptic positive in the sense that $e_\pm = \tau\alpha_\pm + m_\pm \gtrsim \tau + |D'|$. We want at this point to use some natural estimates for first-order factors on the half-lines $\mathbb{R}_\pm$: on $t > 0$ for $\omega \in \mathscr{C}_c^\infty(\mathbb{R})$, λ, γ positive, we compute

$$\begin{aligned}&\|D_t\omega + i(\lambda + \gamma t)\omega\|_{L^2(\mathbb{R}_+)}^2 \\ &\quad= \|D_t\omega\|_{L^2(\mathbb{R}_+)}^2 + \|(\lambda + \gamma t)\omega\|_{L^2(\mathbb{R}_+)}^2 + 2\operatorname{Re}\langle D_t\omega, iH(t)(\lambda + \gamma t)\omega\rangle \\ &\quad\geq \int_0^{+\infty}\Big((\lambda + \gamma t)^2 + \gamma\Big)|\omega(t)|^2 dt + \lambda|\omega(0)|^2 \geq \|\lambda\omega\|_{L^2(\mathbb{R}_+)}^2 + \lambda|\omega(0)|^2,\end{aligned} \tag{1.22}$$

which is somehow a perfect estimate of elliptic type, suggesting that the first-order factor containing e_+ should be easy to handle.

Changing λ in $-\lambda$ gives

$$\begin{aligned}\|D_t\omega + i(-\lambda + \gamma t)\omega\|_{L^2(\mathbb{R}_+)}^2 &\geq 2\operatorname{Re}\langle D_t\omega, iH(t)(-\lambda + \gamma t)\omega\rangle \\ &= \int_0^{+\infty}\gamma|\omega(t)|^2 dt - \lambda|\omega(0)|^2,\end{aligned}$$

so that

$$\|D_t\omega + i(-\lambda + \gamma t)\omega\|_{L^2(\mathbb{R}_+)}^2 + \lambda|\omega(0)|^2 \geq \gamma\|\omega\|_{L^2(\mathbb{R}_+)}^2, \tag{1.23}$$

an estimate of lesser quality, because we need to secure a control of $\omega(0)$ to handle this type of factor.

Now, changing λ in $-\lambda$ and t in $-t$, i.e., working on the negative half-line, yields

$$\|D_t\omega + i(\lambda + \gamma t)\omega\|_{L^2(\mathbb{R}_-)}^2 \geq \|\lambda\omega\|_{L^2(\mathbb{R}_-)}^2 + \lambda|\omega(0)|^2. \tag{1.24}$$

suggesting that the first-order factor containing f_- should be useful in the case $f_- < 0$.

Step 2: case $\mathbf{f_+ \geq 0}$. Looking at formula (1.21), since the factor containing e_+ is elliptic in the sense given above, we have to discuss on the sign of f_+. Identifying the operator with its symbol, we have $f_+ = \tau(\alpha_+ + \beta x_n) - m_+(\xi')$, and thus $\tau\alpha_+ \geq m_+(\xi')$ yields a positive f_+. Iterating the method outlined above on the half-line $\mathbb{R}_+$, we get a nice estimate of the form of (1.22) on $\mathbb{R}_+$; in particular we obtain a control of $v_+(0)$. From the transmission condition, we have $v_+(0) = v_-(0)$ and hence this amounts to also controlling $v_-(0)$. That control along with the natural estimates on $\mathbb{R}_-$ are enough to prove an inequality of the form of the sought Carleman estimate.

Step 3: case $\mathbf{f_+ < 0}$. Here, we assume that $\tau\alpha_+ < m_+(\xi')$. We can still use on $\mathbb{R}_+$ the factor containing e_+, and by (1.21) and (1.22) control the following quantity

$$c_n^+(D_n + if_+)v_+(0) = \overbrace{c_n^+(D_n v_+ + i\tau\alpha_+)v_+(0)}^{=\mathcal{V}_+} - c_n^+ i m_+ v_+(0). \tag{1.25}$$

Our key assumption is

$$f_+(0) < 0 \Longrightarrow f_-(0) \leq 0. \tag{1.26}$$

Under that hypothesis, we can use the negative factor f_- on $\mathbb{R}_-$ (note that f_- is increasing with x_n, so that $f_-(0) \leq 0 \Longrightarrow f_-(x_n) < 0$ for $x_n < 0$). With (1.24) we then control

$$c_n^-(D_n + ie_-)v_-(0) = \underbrace{c_n^-(D_n v_- + i\tau\alpha_-)v_-(0)}_{=\mathcal{V}_-} + c_n^- i m_- v_-(0). \tag{1.27}$$

Because of (1.23) we conclude that nothing more can be achieved with inequalities on each side of the interface. At this point we however notice that the second transmission condition in (1.20) implies $\mathcal{V}_- = \mathcal{V}_+$, yielding the control of the difference of (1.27) and (1.25), i.e., of

$$c_n^- i m_- v_-(0) + c_n^+ i m_+ v_+(0) = i\Big(c_n^- m_- + c_n^+ m_+\Big)v(0).$$

Now, as $c_n^- m_- + c_n^+ m_+$ is elliptic positive, this gives a control of $v(0)$ in (tangential) H^1-norm, which is enough then to get an estimates on both sides that leads to the sought Carleman estimates.

Step 4: stitching estimates together. The analysis we have sketched relies on a separation into two zones in the (τ, ξ') space. Patching the estimates of the form of (1.13) in each zone together allows us to conclude the proof of the Carleman estimate.

1.5. Explaining the key assumption

In the first place, our key assumption, condition (1.26), can be reformulated as

$$\forall \xi' \in \mathbb{S}^{n-2}, \quad \frac{\alpha_+}{\alpha_-} \geq \frac{m_+(\xi')}{m_-(\xi')}. \tag{1.28}$$

In fact[4], (1.26) means $\tau\alpha_+ < m_+(\xi') \Longrightarrow \tau\alpha_- \le m_-(\xi')$ and since $\alpha_\pm, m_\pm$ are all positive, this is equivalent to having $m_+(\xi')/\alpha_+ \le m_-(\xi')/\alpha_-$, which is (1.28). An analogy with an estimate for a first-order factor may shed some light on this condition. With

$$f(t) = H(t)(\tau\alpha_+ + \beta t - m_+) + H(-t)(\tau\alpha_- + \beta t - m_-), \quad \tau, \alpha_\pm, \beta, m_\pm \text{ positive constants,}$$

we want to prove an injectivity estimate of the type $\|D_t v + i f(t) v\|_{L^2(\mathbb{R})} \gtrsim \|v\|_{L^2(\mathbb{R})}$, say for $v \in \mathscr{C}_c^\infty(\mathbb{R})$. It is a classical fact (see e.g. Lemma 3.1.1 in [16]) that such an estimate (for a smooth f) is equivalent to the condition that $t \mapsto f(t)$ does not change sign from $+$ to $-$ while t increases: it means that the adjoint operator $D_t - if(t)$ satisfies the so-called condition (Ψ). Looking at the function f, we see that it increases on each half-line $\mathbb{R}_\pm$, so that the only place to get a “forbidden” change of sign from $+$ to $-$ is at $t = 0$: to obtain an injectivity estimate, we have to avoid the situation where $f(0^+) < 0$ and $f(0^-) > 0$, that is, we have to make sure that $f(0^+) < 0 \Longrightarrow f(0^-) \le 0$, which is indeed the condition (1.28). The function f is increasing affine on $\mathbb{R}_\pm$ with the same slope β on both sides, with a possible discontinuity at 0.

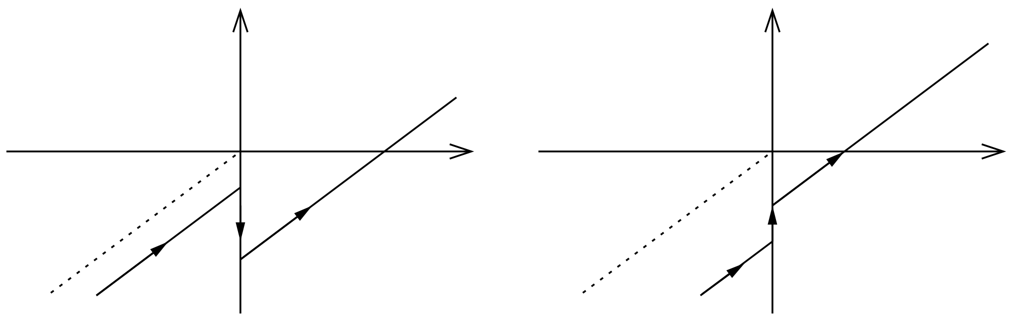

Figure 1.1: $f(0^-) \le 0$; $f(0^+) < 0$.

When $f(0^+) < 0$ we should have $f(0^-) \le 0$ and the line on the left cannot go above the dotted line, in such a way that the discontinuous zigzag curve with the arrows in Figure 1.1 has only a change of sign from $-$ to $+$.

When $f(0^+) \ge 0$, there is no other constraint on $f(0^-)$: even with a discontinuity, the change of sign can only occur from $-$ to $+$.

We prove below (Section 5) that condition (1.28) is relevant to our problem in the sense that it is indeed necessary to have a Carleman estimate with this weight: should (1.28) be violated, we would be able for this model to construct a quasi-mode for $\mathcal{L}_\tau$, i.e. a τ-family of functions v with L^2-norm 1 such that $\|\mathcal{L}_\tau v\|_{L^2} \ll \|v\|_{L^2}$, as

[4]For the main theorem, we shall in fact require the stronger strict inequality

$$\frac{\alpha_+}{\alpha_-} > \frac{m_+(\xi')}{m_-(\xi')}. \tag{1.29}$$

However, we shall see in Section 5 that in the particular case presented here, where the matrix A is piecewise constant and the weight function φ solely depends on x_n the inequality (1.28) is actually a *necessary and sufficient* condition to obtain a Carleman estimate with weight φ.

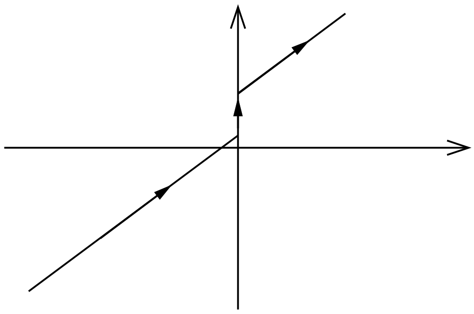

Figure 1.2: $f(0^-) \gtreqless 0$; $f(0^+) \geq 0$.

τ goes to ∞, ruining any hope to prove a Carleman estimate. As usual for this type of construction, it uses some type of complex geometrical optics method, which is easy in this case to implement directly, due to the simplicity of the expression of the operator.

Remark 1.3. A very particular case of anisotropic medium was tackled in [14] for the purpose of proving a controllability result for linear parabolic equations. The condition imposed on the weight function in [14] (Assumption 2.1 therein) is much more demanding than what we impose here. In the isotropic case, $c_j^\pm = c_\pm$ for all $j \in \{1, \ldots, n\}$, we have $m_+ = m_- = |\xi'|$ and our condition (1.29) reads $\alpha_+ > \alpha_-$. Note also that the isotropic case $c_- \geq c_+$ was already considered in [6].

2. Framework

2.1. Preliminaries

Let $\Omega, \Sigma, \mathcal{L}, \mathcal{W}, \varphi$ be as in (1.3), (1.5), (1.11) and (1.12). Let $\mathcal{W}_0 = \{w \in \mathcal{W}, \operatorname{supp} w \text{ compact}\}$. For $\tau \geq 0$ we define the vector space

$$\mathcal{W}_\tau = \{e^{\tau\varphi} w\}_{w \in \mathcal{W}_0} \tag{2.1}$$

For $v \in \mathcal{W}_\tau$, we have $v = e^{\tau\varphi} w$ with $w \in \mathcal{W}_0$ so that, using the notation introduced in (1.4), (1.7), with $v_\pm = e^{\tau\varphi_\pm} w_\pm$, we have

$$v = H_- v_- + H_+ v_+, \tag{2.2}$$

and we see that the transmission conditions (1.8)–(1.9) on w read for v as

$$\begin{cases} v_+ = v_- \text{ at } \Sigma, \\ \langle dv_+ - \tau v_+ d\varphi_+, A_+\nu\rangle = \langle dv_- - \tau v_- d\varphi_-, A_-\nu\rangle \qquad \text{at } \Sigma. \end{cases} \tag{2.3}$$

Observing that $e^{\tau\varphi_\pm} D e^{-\tau\varphi_\pm} = D + i\tau d\varphi_\pm$, for $w \in \mathcal{W}$, we obtain from (1.10),

$$\begin{aligned} &e^{\tau\varphi}\mathcal{L}w \\ &= H_- e^{\tau\varphi_-} D \cdot A_- D w_- + H_+ e^{\tau\varphi_+} D \cdot A_+ D w_+ \\ &= H_- e^{\tau\varphi_-} D \cdot A_- D e^{-\tau\varphi_-} v_- + H_+ e^{\tau\varphi_+} D \cdot A_+ D e^{-\tau\varphi_+} v_+ \\ &= H_-(D + i\tau d\varphi_-) \cdot A_-(D + i\tau d\varphi_-) v_- + H_+(D + i\tau d\varphi_+) \cdot A_+(D + i\tau d\varphi_+) v_+, \end{aligned}$$

so that,

$$\|e^{\tau\varphi}\mathcal{L}w\|^2_{L^2(\mathbb{R}^n)} = \|e^{\tau\varphi}\mathcal{L}e^{-\tau\varphi}v\|^2_{L^2(\mathbb{R}^n)} = \int_{\Omega_-} |(D + i\tau d\varphi_-) \cdot A_-(D + i\tau d\varphi_-) v_-|^2 dx + \int_{\Omega_+} |(D + i\tau d\varphi_+) \cdot A_+(D + i\tau d\varphi_+) v_+|^2 dx. \tag{2.4}$$

Summing-up, with

$$\Xi = \{\text{positive-definite } n \times n \text{ matrices}\},$$

we consider $A_\pm \in \mathscr{C}^\infty(\Omega; \Xi)$, $\varphi_\pm \in \mathscr{C}^\infty(\Omega), v_\pm \in \mathscr{C}^\infty_c(\Omega)$ such that

$$v_+ = v_-, \quad \varphi_+ = \varphi_-, \quad \langle dv_- - \tau d\varphi_-, A_-\nu\rangle = \langle dv_+ - \tau d\varphi_+, A_+\nu\rangle \quad \text{at } \Sigma. \tag{2.5}$$

We define

$$\mathcal{P}_\pm = (D + i\tau d\varphi_\pm) \cdot A_\pm(D + i\tau d\varphi_\pm). \tag{2.6}$$

From the identity (2.4), proving Theorem 1.1 amounts to finding a proper lower-bound for the quantity

$$\|H_-\mathcal{P}_- v_-\|^2_{L^2(\mathbb{R}^n)} + \|H_+\mathcal{P}_+ v_+\|^2_{L^2(\mathbb{R}^n)} = \|e^{\tau\varphi}\mathcal{L}e^{-\tau\varphi}v\|^2_{L^2(\mathbb{R}^n)}. \tag{2.7}$$

2.2. Description in local coordinates

Assuming as we may that the hypersurface Σ is given locally by the equation $\{x_n = 0\}$, we have, using the Einstein convention on repeated indices $j \in \{1, \dots, n-1\}$, and noting from the ellipticity condition that $a_{nn} > 0$ (the matrix $A(x) = (a_{jk}(x))_{1\le j,k\le n}$),

$$\begin{aligned} \mathcal{L} &= D_n a_{nn} D_n + D_n a_{nj} D_j + D_j a_{jn} D_n + D_j a_{jk} D_k, \\ &= D_n a_{nn}\Big(D_n + a_{nn}^{-1} a_{nj} D_j\Big) + D_j a_{jn} D_n + D_j a_{jk} D_k, \end{aligned}$$

so that the transmission conditions (1.8)–(1.9) on $x_n = 0$, are

$$w_+ = w_-, \quad a^+_{nn} D_n w_+ + a^+_{nj} D_j w_+ = a^-_{nn} D_n w_- + a^-_{nj} D_j w_-. \tag{2.8}$$

We note also that with

$$T = a_{nn}^{-1} a_{nj} D_j, \tag{2.9}$$

we have

$$\mathcal{L} = \Big(D_n + T^*\big) a_{nn}\Big(D_n + T\Big) - T^* a_{nn} D_n - T^* a_{nn} T + D_j a_{jn} D_n + D_j a_{jk} D_k$$

and since $T^* = D_j a_{nn}^{-1} a_{nj}$, we have $T^* a_{nn} D_n = D_j a_{nj} D_n = D_j a_{jn} D_n$ and

$$\mathcal{L} = \Big(D_n + T^*\big) a_{nn}\Big(D_n + T\Big) + D_j b_{jk} D_k, \tag{2.10}$$

where the $(n-1)\times(n-1)$ matrix (b_{jk}) is positive-definite since with $\xi' = (\xi_1, \dots, \xi_{n-1})$ and $\xi = (\xi', \xi_n)$,

$$\langle B\xi', \xi'\rangle = \sum_{1\le j,k\le n-1} b_{jk}\xi_j\xi_k = \langle A\xi, \xi\rangle,$$

where $a_{nn}\xi_n = -\sum_{1\leq j\leq n-1} a_{nj}\xi_j$. Note also that $b_{jk} = a_{jk} - (a_{nj}a_{nk}/a_{nn})$.

According to (1.10), for $w \in \mathcal{W}$, we have

$$\mathcal{L}w = H_-\Big[(D_n + T_-^*)a_{nn}^-(D_n + T_-)w_- + D_j b_{jk}^- D_k w_-\Big] \\ + H_+\Big[(D_n + T_+^*)a_{nn}^+(D_n + T_+)w_+ + D_j b_{jk}^+ D_k w_+\Big] \quad (2.11)$$

and the transmission conditions (1.8), (1.9) read

$$w_- = w_+, \quad a_{nn}^-(D_n + T_-)w_- = a_{nn}^+(D_n + T_+)w_+, \quad \text{at } \Sigma. \quad (2.12)$$

2.3. Pseudo-differential factorization on each side

For simplicity we consider here the weight function $\varphi = H_+\varphi_+ + H_-\varphi_-$ with $\varphi_\pm$ that solely depend on x_n.

We define for $m \in \mathbb{R}$ the class of tangential standard symbols $\mathcal{S}^m$ as the smooth functions on $\mathbb{R}^n \times \mathbb{R}^{n-1}$ such that, for all $(\alpha, \beta) \in \mathbb{N}^n \times \mathbb{N}^{n-1}$,

$$\sup_{(x,\xi')\in\mathbb{R}^n\times\mathbb{R}^{n-1}} \langle\xi'\rangle^{-m+|\beta|} |(\partial_x^\alpha \partial_{\xi'}^\beta a)(x, \xi')| < \infty, \quad (2.13)$$

with $\langle\xi'\rangle = \Big(1 + |\xi'|^2\Big)^{\frac{1}{2}}$. Some basic properties of standard pseudo-differential operators are recalled in Appendix A.

Section 2.2 and formulæ (2.6), (2.11) gives

$$\mathcal{P}_\pm = \Big(D_n + i\tau\varphi_\pm' + T_\pm^*\Big)a_{nn}^\pm\Big(D_n + i\tau\varphi_\pm' + T_\pm\Big) + D_j b_{jk}^\pm D_k. \quad (2.14)$$

Let $M_\pm \in \mathrm{op}(\mathcal{S}^1)$ be the principal part of a pseudo-differential positive selfadjoint square root of $D_j b_{jk} D_k$. We denote by $m_\pm$ its principal symbol. We have $m_\pm \geq C\langle\xi'\rangle$. For $|\xi'|$ sufficiently large, say $|\xi'| \geq 1$, we have

$$m_\pm = \Big(\frac{b_{jk}^\pm}{a_{nn}^\pm}\xi_j\xi_k\Big)^{\frac{1}{2}}. \quad (2.15)$$

In particular $m_\pm$ is homogeneous of degree one in ξ', for $|\xi'| \geq 1$.

Introduce

$$\Sigma^1 = \mathcal{S}^1 + \tau\mathcal{S}^0 + \mathcal{S}^0\xi_n, \qquad \Psi^1 = \mathrm{op}(\mathcal{S}^1) + \tau\mathrm{op}(\mathcal{S}^0) + \mathrm{op}(\mathcal{S}^0)D_n. \quad (2.16)$$

Modulo the operator class Ψ^1 we may write

$$\mathcal{P}_+ \equiv \mathcal{P}_{E+}a_{nn}^+\mathcal{P}_{F+}, \qquad \mathcal{P}_- \equiv \mathcal{P}_{F-}a_{nn}^-\mathcal{P}_{E-}, \quad (2.17)$$

where

$$\mathcal{P}_{E\pm} = D_n + S_\pm + i(\underbrace{\tau\varphi_\pm' + M_\pm}_{E_\pm}), \quad \mathcal{P}_{F\pm} = D_n + S_\pm + i(\underbrace{\tau\varphi_\pm' - M_\pm}_{F_\pm}), \quad (2.18)$$

with

$$S_\pm = s^w(x, D'), \quad s_\pm = \sum_{1\leq j\leq n-1} \frac{a_{nj}^\pm}{a_{nn}^\pm}\xi_j, \quad \text{so that } S_\pm^* = S_\pm,\ S_\pm = T_\pm + \frac{1}{2}\operatorname{div} T_\pm,$$

where $T_\pm$ is the vector field $\sum_{1\leq j\leq n-1} \frac{a_{nj}^\pm}{ia_{nn}^\pm}\partial_j$.

We denote by $f_\pm$ and $e_\pm$ the symbols of $F_\pm$ and $E_\pm$ respectively, modulo the symbol class Σ^1.

The transmission conditions (2.5) with our choice of coordinates read, at $x_n = 0$,

$$v_- = v_+, \quad a_{nn}^-(D_n + T_- + i\tau\varphi'_-)v_- = a_{nn}^+(D_n + T_+ + i\tau\varphi'_+)v_+. \tag{2.19}$$

Remark 2.1. Note that the Carleman estimate we shall prove is insensitive to terms in Ψ^1 in the conjugated operator $\mathcal{P}$. Formulæ (2.17) and (2.18) for $\mathcal{P}_+$ and $\mathcal{P}_-$ will thus be the base of our analysis.

Remark 2.2. In the articles [14, 15], the zero crossing of the roots of the symbol of $\mathcal{P}_\pm$, as seen as a polynomial in ξ_n, is analysed. Here, the factorization into first-order operators isolates each root. In fact, $f_\pm$ changes sign and we shall impose a condition on the weight function at the interface to obtain a certain scheme for this sign change. See Section 4.

2.4. Choice of weight-function

From (2.14), the symbols of $\mathcal{P}_\pm$, modulo the symbol class Σ^1, are given by $p_\pm(x,\xi,\tau) = a_{nn}^\pm\Big(q_2^\pm + 2iq_1^\pm\Big)$ with

$$q_2^\pm = (\xi_n + s_\pm)^2 + \frac{b_{jk}^\pm}{a_{nn}^\pm}\xi_j\xi_k - \tau^2(\varphi'_\pm)^2, \qquad q_1^\pm = \tau\varphi'_\pm(\xi_n + s_\pm),$$

and from the construction of $m_\pm$, for $|\xi'| \geq 1$, we have

$$q_2^\pm = (\xi_n + s_\pm)^2 + m_\pm^2 - (\tau\varphi'_\pm)^2 = (\xi_n + s_\pm)^2 - f_\pm e_\pm. \tag{2.20}$$

We formulate the usual *sub-ellipticity* condition on the weight function.

Assumption 2.3. The weight function φ is such that

$$q_2^\pm = 0 \text{ and } q_1^\pm = 0 \quad\Longrightarrow\quad \{q_2^\pm, q_1^\pm\} > 0,$$

It is important to note that this property is coordinate free. For elliptic operators with smooth coefficients this property is necessary and sufficient for a Carleman estimate as that of Theorem 1.1 to hold (see [8] or e.g. [12]).

There are various "classical" forms for the weight function φ. For instance, one may use

$$\varphi_\pm(x_n) = \alpha_\pm x_n + \beta x_n^2/2, \quad \alpha_\pm > 0, \quad \beta > 0, \tag{2.21}$$

as already used in the sketch of the proof in the introductory section. One may also use $\varphi(x_n) = e^{\beta\phi(x_n)}$ with the function ϕ of the form

$$\phi = H_-\phi_- + H_+\phi_+, \quad \phi_\pm \in \mathscr{C}_c^\infty(\mathbb{R}),$$

such that ϕ is *continuous* and $|\phi'_\pm| \geq C > 0$. In both cases, Assumption 2.3 can be achieved by choosing the parameter β sufficiently large. In any case we set

$$\alpha_\pm = \varphi'_{|x_n=0^\pm}.$$

Of course, with the present non-smooth coefficient case, we shall impose some conditions on the constants $\alpha_\pm$, depending on the jump discontinuities of the matrix A. This will be exposed in Section 4.

The sub-ellipticity condition above implies the following lemma.

Lemma 2.4. *Let $\lambda^2 = 1 + \tau^2 + |\xi'|^2$. There exist $C > 0$, $C' > 0$ $\tau_1 > 1$ and $\delta > 0$ such that for $\tau \geq \tau_0$*

$$|f_\pm| \leq \delta\lambda \quad\Longrightarrow\quad \tau/C \leq |\xi'| \leq C\tau \;\; \textit{and} \;\; \{\xi_n + s_\pm, f_\pm\} \geq C'\lambda.$$

3. Estimates for first-order factors

Most of the pseudo-differential calculus arguments we use concern the calculus with the large parameter τ presented in Appendix A.2.

The $L^2(\mathbb{R}^n)$ and $L^2(\mathbb{R}^{n-1})$ dot-products will be both denoted by $\langle \cdot, \cdot \rangle$. Sobolev norms with the positive parameter τ are denoted as follows

$$|v|_{\mathcal{H}^\sigma} = \tau^\sigma |v|_{L^2(\mathbb{R}^{n-1})} + |\langle D_{x'} \rangle^\sigma v|_{L^2(\mathbb{R}^{n-1})}, \quad \sigma \geq 0. \tag{3.1}$$

3.1. Positive imaginary part on a half-line

We have the following estimates for the operators $\mathcal{P}_{E+}$ and $\mathcal{P}_{E-}$ given in (2.18). They can be proven with multiplier techniques.

Lemma 3.1. *Let $\ell \geq 0$. There exists $\tau_1 \geq 1$ such that*

$$\|H_+ \mathcal{P}_{E_+} \omega\|_{L^2(\mathbb{R};\mathcal{H}^\ell)} \gtrsim |\omega_{|x_n=0^+}|_{\mathcal{H}^{\ell+\frac{1}{2}}} + \|H_+ \omega\|_{L^2(\mathbb{R};\mathcal{H}^{\ell+1})} + \|H_+ D_n \omega\|_{L^2(\mathbb{R};\mathcal{H}^\ell)}, \tag{3.2}$$

and

$$\|H_- \mathcal{P}_{E_-} \omega\|_{L^2(\mathbb{R};\mathcal{H}^\ell)} + |\omega_{|x_n=0^-}|_{\mathcal{H}^{\ell+\frac{1}{2}}} \gtrsim \|H_- \omega\|_{L^2(\mathbb{R};\mathcal{H}^{\ell+1})} + \|H_- D_n \omega\|_{L^2(\mathbb{R};\mathcal{H}^\ell)}, \tag{3.3}$$

for $\tau \geq \tau_1$ and $\omega \in \mathscr{C}_c^\infty(\mathbb{R}^n)$.

Note that the first estimate, in $\mathbb{R}_+$, is of very good quality as both the trace and the volume norms are dominated: we have a perfect elliptic estimate. In $\mathbb{R}_-$, we obtain an estimate of lesser quality.

For the operator $\mathcal{P}_{F+}$ we can also obtain a microlocal estimate. We place ourselves in a microlocal region where $f_+ = \tau\varphi^+ - m_+$ is positive. More precisely, let $\chi(\tau, \xi') \in \mathcal{S}(\lambda^0)$, a Fourier multiplier, be such that $|\xi'| \leq C\tau$ and $f_+ \geq C_1 \lambda$ in $\operatorname{supp}(\chi)$, $C_1 > 0$.

Lemma 3.2. *Let $\ell \geq 0$. There exists $\tau_1 \geq 1$ such that*

$$\begin{aligned} &\|H_+ \mathcal{P}_{F_+} \operatorname{op}^w(\chi)\omega\|_{L^2(\mathbb{R};\mathcal{H}^\ell)} + \|H_+ \omega\|_{L^2(\mathbb{R}^n)} \\ &\quad \gtrsim |\operatorname{op}^w(\chi)\omega_{|x_n=0^+}|_{\mathcal{H}^{\ell+\frac{1}{2}}} + \|H_+ \operatorname{op}^w(\chi)\omega\|_{L^2(\mathbb{R};\mathcal{H}^{\ell+1})}, + \|H_+ D_n \operatorname{op}^w(\chi)\omega\|_{L^2(\mathbb{R};\mathcal{H}^\ell)}, \end{aligned}$$

for $\tau \geq \tau_1$ and $\omega \in \mathscr{C}_c^\infty(\mathbb{R}^n)$.

As for (3.2) of Lemma 3.1, up to an harmless remainder term, we thus obtain an elliptic estimate in this microlocal region.

3.2. Negative imaginary part on the negative half-line

Here we place ourselves in a microlocal region where $f_- = \tau\varphi^- - m_-$ is negative. More precisely, let $\chi(\tau, \xi') \in \mathcal{S}(\lambda^0)$, a Fourier multiplier, be such that $|\xi'| \geq C\tau$ and $f_- \leq -C_1 \lambda$ in $\operatorname{supp}(\chi)$, $C_1 > 0$. We have the following lemma whose form is adapted to our needs in the next section. Up to harmless remainder terms, this can also be considered as an elliptic estimate.

Lemma 3.3. *There exists $\tau_1 \geq 1$ such that*

$$\|H_- \mathcal{P}_{F_-} u\|_{L^2(\mathbb{R}^n)} + \|H_- \omega\|_{L^2(\mathbb{R}^n)} + \|H_- D_n \omega\|_{L^2(\mathbb{R}^n)} \gtrsim |u_{|x_n=0^-}|_{\mathcal{H}^{\frac{1}{2}}} + \|H_- u\|_{L^2(\mathbb{R};\mathcal{H}^1)}, \tag{3.4}$$

for $\tau \geq \tau_1$ and $u = a_{nn}^- \mathcal{P}_{E-} \operatorname{op}^w(\chi)\omega$ with $\omega \in \mathscr{C}_c^\infty(\mathbb{R}^n)$.

3.3. Increasing imaginary part on a half-line

The following estimate can be proven by classical techniques for Carleman estimates using Lemma 2.4, i.e., the sub-ellipticity property of the weight function. This estimate exhibits a loss of a half derivative.

Lemma 3.4. *There exists $\tau_1 \geq 1$ such that*

$$\|H_\pm \mathcal{P}_{F_\pm} \omega\|_{L^2(\mathbb{R};\mathcal{H}^\ell)} + |\omega_{|x_n=0^\pm}|_{\mathcal{H}^{\ell+\frac{1}{2}}} \gtrsim \tau^{-\frac{1}{2}} \Big(\|H_\pm \omega\|_{L^2(\mathbb{R};\mathcal{H}^{\ell+1})} + \|H_\pm D_n \omega\|_{L^2(\mathbb{R};\mathcal{H}^\ell)} \Big),$$

for $\tau \geq \tau_1$ and $\omega \in \mathscr{C}_c^\infty(\mathbb{R}^n)$.

Remark 3.5. Observe that in any region where $\tau \gtrsim |\xi'|$ the operator $M_\pm \in \mathrm{op}(\mathcal{S}^1)$ is not in the calculus with the large parameter. This can create difficulties in some of the proofs of the estimates below. This technical point can be circumvented by introducing a cut-off function in (τ, ξ'). These details are omitted here. They can be found in [13].

4. Proof of the Carleman estimate

4.1. The geometric hypothesis

The operators $M_\pm$ have a principal symbol $m_\pm(x,\xi')$ in $\mathcal{S}^1$, which is positively-homogeneous[5] of degree 1 and elliptic, i.e. there exists $\lambda_0^\pm, \lambda_1^\pm$ positive such that for $|\xi'| \geq 1, x \in \mathbb{R}^n$,

$$\lambda_0^\pm |\xi'| \leq m_\pm(x,\xi') \leq \lambda_1^\pm |\xi'|. \tag{4.1}$$

Our main assumption is the following.

Assumption 4.1. The weight function φ is chosen such that

$$\frac{\alpha_+}{\alpha_-} > \frac{\lambda_1^+}{\lambda_0^-}, \qquad \alpha_\pm = \partial_{x_n} \varphi_{\pm|x_n=0^\pm}. \tag{4.2}$$

Let us explain the immediate consequences of that assumption: first of all, we can reformulate it by saying that

$$\exists \sigma > 1, \quad \frac{\alpha_+}{\alpha_-} = \sigma^2 \frac{\lambda_1^+}{\lambda_0^-}. \tag{4.3}$$

Let $1 < \sigma_0 < \sigma$.

First, consider $(\xi', \tau) \in \mathbb{R}^{n-1} \times \mathbb{R}^{+,*}$ such that

$$\tau\alpha_+ \geq \sigma_0 \lambda_1^+ |\xi'|. \tag{4.4}$$

Observe that we then have

$$\begin{aligned} \tau\alpha_+ - m_+(x,\xi') \geq \tau\alpha_+ - \lambda_1^+|\xi'| &\geq \tau\alpha_+(1-\sigma_0^{-1}) \\ &\geq \frac{\sigma_0 - 1}{2\sigma_0}\tau\alpha_+ + \frac{\sigma_0-1}{2}\lambda_1^+|\xi'| \geq C\lambda. \end{aligned}$$

As $f_+ = \tau(\varphi' - \alpha_+) + \tau\alpha_+ - m_+(x,\xi')$, for the support of v_+ sufficiently small, we obtain $f_+ \geq C\lambda$, which means that f_+ is elliptic positive in that region.

Second, if we now have

$$\tau\alpha_+ \leq \sigma\lambda_1^+|\xi'|, \tag{4.5}$$

[5] The homogeneity property means as usual $m_\pm(x,\rho\xi') = \rho m_\pm(x,\xi')$ for $\rho \geq 1, |\xi'| \geq 1$.

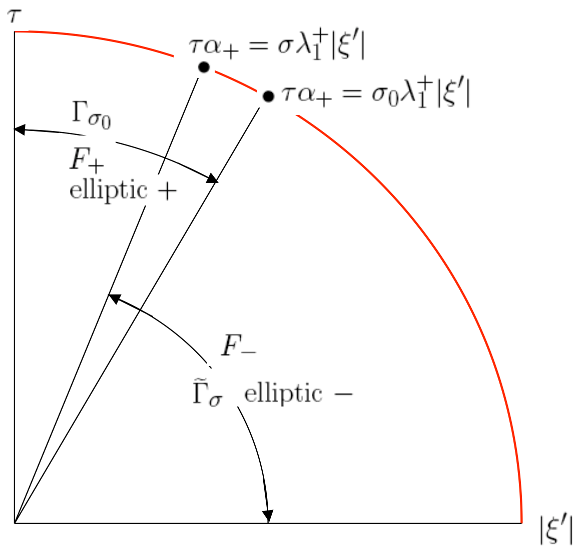

Figure 4.1: The overlapping cones $\Gamma_{\sigma_0}, \widetilde{\Gamma}_\sigma$ in the $\tau, |\xi'|$ plane.

we get (note that $\xi' \neq 0$ since $\tau > 0$) that $\tau\alpha_- \leq \sigma^{-1}\lambda_0^-|\xi'|$: otherwise we would have $\tau\alpha_- > \sigma^{-1}\lambda_0^-|\xi'|$ and thus

$$\frac{\lambda_0^-|\xi'|}{\sigma\alpha_-} < \tau \leq \frac{\sigma\lambda_1^+|\xi'|}{\alpha_+} \Longrightarrow \frac{\alpha_+}{\alpha_-} < \sigma^2\frac{\lambda_1^+}{\lambda_0^-} = \frac{\alpha_+}{\alpha_-} \quad \text{which is impossible.}$$

As a consequence we have

$$\tau\alpha_- - m_-(x,\xi') \leq \tau\alpha_- - \lambda_0^-|\xi'| \leq -\lambda_0^-|\xi'|\frac{(\sigma-1)}{\sigma}$$
$$\leq -\lambda_0^-|\xi'|\frac{(\sigma-1)}{2\sigma} - \frac{(\sigma-1)}{2}\tau\alpha_- \leq -C\lambda. \quad (4.6)$$

With $f_- = \tau(\varphi' - \alpha_-) + \tau\alpha_- - m_-(x,\xi')$, for the support of v_- sufficiently small, we obtain $f_- \leq -C\lambda$, which means that f_- is elliptic negative in that region.

We have thus proven the following result.

Lemma 4.2. *Let $\sigma > \sigma_0 > 1$, and $\alpha_\pm, \lambda_1^+, \lambda_0^+$ be positive numbers such that* (4.3) *holds. For $s > 0$, we define the following cones in $\mathbb{R}^{n-1}_{\xi'} \times \mathbb{R}^*_+$ by*

$$\Gamma_s \equiv \tau\alpha_+ > s\lambda_1^+|\xi'|, \quad \widetilde{\Gamma}_s \equiv \tau\alpha_+ < s\lambda_1^+|\xi'|.$$

For the compact set K sufficiently small, we have $\mathbb{R}^{n-1} \times \mathbb{R}^*_+ = \Gamma_{\sigma_0} \cup \widetilde{\Gamma}_\sigma$ and

$$\Gamma_{\sigma_0} \subset \Big\{(\xi',\tau) \in \mathbb{R}^{n-1} \times \mathbb{R}^*_+, \forall x_n \geq 0, x \in \operatorname{supp}(v_+) \subset K,\ f_+(x,\xi') \geq C\lambda\Big\}, \quad (4.7)$$
$$\widetilde{\Gamma}_\sigma \subset \Big\{(\xi',\tau) \in \mathbb{R}^{n-1} \times \mathbb{R}^*_+, \forall x_n \leq 0,, x \in \operatorname{supp}(v_-) \subset K, f_-(x,\xi') \leq -C\lambda\Big\}. \quad (4.8)$$

N.B. The key result for the sequel is that Assumption (4.2) is securing the fact that the overlapping open cones $\Gamma_{\sigma_0}, \widetilde{\Gamma}_\sigma$ are such that on Γ_{σ_0}, f_+ is elliptic positive and on $\widetilde{\Gamma}_\sigma$, f_- is elliptic negative (for the calculus with a large parameter of Appendix A.2).

Using a partition of unity and symbolic calculus, we shall be able to assume that either F_+ is elliptic positive, or F_- is elliptic negative.

With the two overlapping cones, for $\tau \geq \tau_2$, we introduce an homogeneous partition of unity

$$1 = \chi_0(\xi', \tau) + \chi_1(\xi', \tau), \quad \underbrace{\operatorname{supp}(\chi_0) \subset \Gamma_{\sigma_0}}_{|\xi'| \lesssim \tau,\ f_+ \text{ elliptic } > 0}, \qquad \underbrace{\operatorname{supp}(\chi_1) \subset \widetilde{\Gamma}_\sigma}_{|\xi'| \gtrsim \tau,\ f_- \text{ elliptic } < 0}. \tag{4.9}$$

We have $\chi_0,\ \chi_1 \in \mathcal{S}(\lambda^0)$. with these Fourier multipliers we associate the following operators.

$$\Xi_j = \operatorname{op}(\chi_j),\ j = 0, 1 \text{ and we have } \Xi_0^2 + \Xi_1^2 = \operatorname{Id}. \tag{4.10}$$

From the transmission conditions (2.19) we find

$$\Xi_j v_{-|x_n=0^-} = \Xi_j v_{+|x_n=0^+} = \Xi_j v_{|x_n=0^+}, \tag{4.11}$$

and

$$a_{nn}^-(D_n + T_- + i\tau\varphi'_-)\Xi_j v_{-|x_n=0^-} = a_{nn}^+(D_n + T_+ + i\tau\varphi'_+)\Xi_j v_{+|x_n=0^+} + \operatorname{op}^w(\kappa_0) v_{|x_n=0^+}, \quad j = 0, 1,$$

with $\kappa_0 \in \mathcal{S}(\lambda^0)$, that originates from commutators. Defining $\mathcal{V}_{j,\pm} = a_{nn}^\pm(D_n + S_\pm + i\tau\varphi'_\pm)\Xi_j v_{\pm|x_n=0^\pm}$ we find

$$\mathcal{V}_{j,-} = \mathcal{V}_{j,+} + \operatorname{op}^w(\kappa_1) v_{|x_n=0^+}, \quad \kappa_1 \in \mathcal{S}(\lambda^0). \tag{4.12}$$

We shall now prove microlocal Carleman estimates in the two overlapping regions, Γ_{σ_0} and $\tilde{\Gamma}_\sigma$

4.2. Region Γ_{σ_0}: both roots are positive on the positive half-line

On the one hand, From Lemma 3.1 we have

$$\|H_+\mathcal{P}_+\Xi_0 v_+\|_{L^2(\mathbb{R}^n)} \gtrsim |\mathcal{V}_{0,+} - ia_{nn}^+ M_+ \Xi_0 v_{+|x_n=0^+}|_{\mathcal{H}^{\frac12}} + \|H_+\mathcal{P}_{F+}\Xi_0 v_+\|_{L^2(\mathbb{R};\mathcal{H}^1)}. \tag{4.13}$$

The ellipticity of F_+ on the support of χ_0 allows us to reiterate the estimate by Lemma 3.2 to obtain

$$\|H_+\mathcal{P}_+\Xi_0 v_+\|_{L^2(\mathbb{R}^n)} + \|H_+ v_+\|_{L^2(\mathbb{R}^n)} \gtrsim |\mathcal{V}_{0,+} - ia_{nn}^+ M_+ \Xi_0 v_{+|x_n=0^+}|_{\mathcal{H}^{\frac12}} + |\Xi_0 v_{+|x_n=0^+}|_{\mathcal{H}^{\frac32}} + \|H_+\Xi_0 v_+\|_{L^2(\mathbb{R},\mathcal{H}^2)} + \|H_+ D_n \Xi_0 v_+\|_{L^2(\mathbb{R},\mathcal{H}^1)}.$$

Since we have also

$$|\mathcal{V}_{0,+}|_{\mathcal{H}^{\frac12}} \lesssim |\mathcal{V}_{0,+} - ia_{nn}^+ M_+ \Xi_0 v_{+|x_n=0^+}|_{\mathcal{H}^{\frac12}} + |\Xi_0 v_{+|x_n=0^+}|_{\mathcal{H}^{\frac32}}, \tag{4.14}$$

we obtain

$$\|H_+\mathcal{P}_+\Xi_0 v_+\|_{L^2(\mathbb{R}^n)} + \|H_+ v_+\|_{L^2(\mathbb{R}^n)} \gtrsim |\mathcal{V}_{0,+}|_{\mathcal{H}^{\frac12}} + |\Xi_0 v_{+|x_n=0^+}|_{\mathcal{H}^{\frac32}} + \|H_+\Xi_0 v_+\|_{L^2(\mathbb{R},\mathcal{H}^2)} + \|H_+ D_n \Xi_0 v_+\|_{L^2(\mathbb{R},\mathcal{H}^1)}. \tag{4.15}$$

On the other hand, with Lemma 3.4 we have

$$\|H_-\mathcal{P}_-\Xi_0 v_-\|_{L^2(\mathbb{R}^n)} + |\mathcal{V}_{0,-} + ia_{nn}^- M_- \Xi_0 v_{-|x_n=0^-}|_{\mathcal{H}^{\frac12}} \gtrsim \tau^{-\frac12}\|H_-\mathcal{P}_{E-}\Xi_0 v_-\|_{L^2(\mathbb{R};\mathcal{H}^1)},$$

which with Lemma 3.1 yields

$$\|H_-\mathcal{P}_-\Xi_0 v_-\|_{L^2(\mathbb{R}^n)} + |\mathcal{V}_{0,-} + ia^-_{nn}M_-\Xi_0 v_{-|x_n=0^-}|_{\mathcal{H}^{\frac12}} + \tau^{-\frac12}|\Xi_0 v_{-|x_n=0^-}|_{\mathcal{H}^{\frac32}} \gtrsim \tau^{-\frac12}\Big(\|H_-\Xi_0 v_-\|_{L^2(\mathbb{R};\mathcal{H}^2)} + \|H_-\Xi_0 D_n v_-\|_{L^2(\mathbb{R};\mathcal{H}^1)}\Big).$$

Arguing as for (4.14) we find

$$\|H_-\mathcal{P}_-\Xi_0 v_-\|_{L^2(\mathbb{R}^n)} + |\mathcal{V}_{0,-}|_{\mathcal{H}^{\frac12}} + |\Xi_0 v_{-|x_n=0^-}|_{\mathcal{H}^{\frac32}} \gtrsim \tau^{-\frac12}\Big(\|H_-\Xi_0 v_-\|_{L^2(\mathbb{R};\mathcal{H}^2)} + \|H_-\Xi_0 D_n v_-\|_{L^2(\mathbb{R};\mathcal{H}^1)}\Big). \quad (4.16)$$

Now, from the transmission conditions (4.11)–(4.12), by summing $\varepsilon(4.16)+(4.15)$ we obtain

$$\|H_-\mathcal{P}_-\Xi_0 v_-\|_{L^2(\mathbb{R}^n)} + \|H_+\mathcal{P}_+\Xi_0 v_+\|_{L^2(\mathbb{R}^n)} + \|H_+ v_+\|_{L^2(\mathbb{R}^n)} \gtrsim |\mathcal{V}_{0,-}|_{\mathcal{H}^{\frac12}} + |\mathcal{V}_{0,+}|_{\mathcal{H}^{\frac12}} + |\Xi_0 v_{|x_n=0}|_{\mathcal{H}^{\frac32}} + \tau^{-\frac12}\Big(\|\Xi_0 v\|_{L^2(\mathbb{R};\mathcal{H}^2)} + \|H_-\Xi_0 D_n v_-\|_{L^2(\mathbb{R};\mathcal{H}^1)} + \|H_+\Xi_0 D_n v_+\|_{L^2(\mathbb{R};\mathcal{H}^1)}\Big).$$

by choosing $\varepsilon > 0$ sufficiently small and τ sufficiently large. Finally, recalling the form of $\mathcal{V}_{0,\pm}$ we obtain

$$\|H_-\mathcal{P}_-\Xi_0 v_-\|_{L^2(\mathbb{R}^n)} + \|H_+\mathcal{P}_+\Xi_0 v_+\|_{L^2(\mathbb{R}^n)} + \|H_+ v_+\|_{L^2(\mathbb{R}^n)} \gtrsim |\Xi_0 D_n v_{-|x_n=0^-}|_{\mathcal{H}^{\frac12}}|\Xi_0 D_n v_{+|x_n=0^+}|_{\mathcal{H}^{\frac12}} + |\Xi_0 v_{|x_n=0}|_{\mathcal{H}^{\frac32}} + \tau^{-\frac12}\Big(\|\Xi_0 v\|_{L^2(\mathbb{R};\mathcal{H}^2)} + \|H_-\Xi_0 D_n v_-\|_{L^2(\mathbb{R};\mathcal{H}^1)} + +\|H_+\Xi_0 D_n v_+\|_{L^2(\mathbb{R};\mathcal{H}^1)}\Big). \quad (4.17)$$

4.3. Region $\tilde{\Gamma}_\sigma$: only one root is positive on the positive half-line

This case is more difficult a priori since we cannot expect to control $v_{|x_n=0^+}$ directly from the estimates of the first-order factors. Nevertheless when the positive ellipticity of F_+ is violated, the complement of $\Gamma^+_{\sigma_0}$ is included in Γ^-_σ in which F_- is elliptic negative: this is the result of our main geometric assumption in Lemma 4.2.

As in (4.13) we have

$$\|H_+\mathcal{P}_+\Xi_1 v_+\|_{L^2(\mathbb{R}^n)} \gtrsim |\mathcal{V}_{1,+} - ia^+_{nn}M_+\Xi_1 v_{+|x_n=0^+}|_{\mathcal{H}^{\frac12}} + \|H_+\mathcal{P}_{F+}\Xi_1 v_+\|_{L^2(\mathbb{R};\mathcal{H}^1)}.$$

and using Lemma 3.3 for the negative half-line, we have

$$\|H_-\mathcal{P}_-\Xi_1 v_-\|_{L^2(\mathbb{R}^n)} + \|H_- v_-\|_{L^2(\mathbb{R}^n)} + \|H_- D_n v_-\|_{L^2(\mathbb{R}^n)} \gtrsim |\mathcal{V}_{1,-} + ia^-_{nn}M_-\Xi_1 v_-|_{|x_n=0^-}|_{\mathcal{H}^{\frac12}} + \|H_-\mathcal{P}_{E-}\Xi_1 v_-\|_{L^2(\mathbb{R};\mathcal{H}^1)},$$

A quick glance at the above estimate shows that none could be iterated in a favorable manner, since F_+ could be negative on the positive half-line and E_- is indeed positive on the negative half-line. We have to use the additional information given by the transmission conditions. From the above inequalities, we control

$$|\mathcal{V}_{1,-} + ia^-_{nn}M_-\Xi_1 v_{-|x_n=0^-}|_{\mathcal{H}^{\frac12}} + |-\mathcal{V}_{1,+} + ia^+_{nn}M_+\Xi_1 v_{+|x_n=0^+}|_{\mathcal{H}^{\frac12}}$$

which, by transmission conditions (4.11)–(4.12) implies the control of

$$\begin{aligned}&|\mathcal{V}_{1,-} - \mathcal{V}_{1,+} i a_{nn}^{-} M_{-} \Xi_1 v_{-|x_n=0^-} + i a_{nn}^{+} M_{+} \Xi_1 v_{+|x_n=0^+}|_{\mathcal{H}^{\frac12}}\\ &\quad = |\Big(\mathrm{op}^w(\kappa_1) + a_{nn}^{-} M_{-} + a_{nn}^{+} M_{+}\Big) \Xi_1 v_{|x_n=0}|_{\mathcal{H}^{\frac12}}\end{aligned}$$

For τ sufficiently large, the positivity of $m_\pm$ yields

$$\begin{aligned}&|\mathcal{V}_{1,-} + i a_{nn}^{-} M_{-} \Xi_1 v_{-|x_n=0^-}|_{\mathcal{H}^{\frac12}} + |-\mathcal{V}_{1,+} + i a_{nn}^{+} M_{+} \Xi_1 v_{+|x_n=0^+}|_{\mathcal{H}^{\frac12}}\\ &\quad + |v_{|x_n=0}|_{L^2(\mathbb{R}^{n-1})} \gtrsim |\Xi_1 v_{|x_n=0}|_{\mathcal{H}^{\frac32}} + |\Xi_1 D_n v_{-|x_n=0^-}|_{\mathcal{H}^{\frac12}} + |\Xi_1 D_n v_{+|x_n=0^+}|_{\mathcal{H}^{\frac12}}.\end{aligned}$$

We thus have

$$\begin{aligned}&\|H_{-}\mathcal{P}_{-}\Xi_1 v_{-}\|_{L^2(\mathbb{R}^n)} + \|H_{+}\mathcal{P}_{+}\Xi_1 v_{+}\|_{L^2(\mathbb{R}^n)}\\ &\quad + \|H_{-} v_{-}\|_{L^2(\mathbb{R}^n)} + \|H_{-} D_n v_{-}\|_{L^2(\mathbb{R}^n)} + |v_{|x_n=0}|_{L^2(\mathbb{R}^{n-1})}\\ &\qquad \gtrsim |\Xi_1 v_{|x_n=0}|_{\mathcal{H}^{\frac32}} + |\Xi_1 D_n v_{-|x_n=0^-}|_{\mathcal{H}^{\frac12}} + |\Xi_1 D_n v_{+|x_n=0^+}|_{\mathcal{H}^{\frac12}}\\ &\qquad\quad + \|H_{-}\mathcal{P}_{E-}\Xi_1 v_{-}\|_{L^2(\mathbb{R};\mathcal{H}^1)} + \|H_{+}\mathcal{P}_{F+}\Xi_1 v_{+}\|_{L^2(\mathbb{R};\mathcal{H}^1)}.\end{aligned}$$

The remaining part of the discussion is very similar to the last part of the argument in the previous subsection. By Lemmata 3.1 and 3.4 we have

$$\|H_{-}\mathcal{P}_{E-}\Xi_1 v_{-}\|_{L^2(\mathbb{R};\mathcal{H}^1)} + |\Xi_1 v_{|x_n=0}|_{\mathcal{H}^{\frac32}} \gtrsim \|H_{-}\Xi_1 v_{-}\|_{L^2(\mathbb{R};\mathcal{H}^2)} + \|H_{-}\Xi_1 D_n v_{-}\|_{L^2(\mathbb{R};\mathcal{H}^1)}$$

and

$$\begin{aligned}&\|H_{+}\mathcal{P}_{F_+}\Xi_1 v_{+}\|_{L^2(\mathbb{R};\mathcal{H}^1)} + |\Xi_1 v_{|x_n=0}|_{\mathcal{H}^{\frac32}}\\ &\qquad \gtrsim \tau^{-\frac12}\Big(\|H_{+}\Xi_1 v_{+}\|_{L^2(\mathbb{R};\mathcal{H}^2)} + \|H_{+}\Xi_1 D_n v_{+}\|_{L^2(\mathbb{R};\mathcal{H}^1)}\Big).\end{aligned}$$

Since $|\Xi_1 v_{|x_n=0}|_{\mathcal{H}^{\frac32}}$ is already controlled, we control as well the r.h.s. of the above inequalities and have

$$\begin{aligned}&\|H_{-}\mathcal{P}_{-}\Xi_1 v_{-}\|_{L^2(\mathbb{R}^n)} + \|H_{+}\mathcal{P}_{+}\Xi_1 v_{+}\|_{L^2(\mathbb{R}^n)}\\ &\quad + \|H_{-} v_{-}\|_{L^2(\mathbb{R}^n)} + \|H_{-} D_n v_{-}\|_{L^2(\mathbb{R}^n)} + |v_{|x_n=0}|_{L^2(\mathbb{R}^{n-1})}\\ &\qquad \gtrsim |\Xi_1 v_{|x_n=0}|_{\mathcal{H}^{\frac32}} + |\Xi_1 D_n v_{-|x_n=0^-}|_{\mathcal{H}^{\frac12}} + |\Xi_1 D_n v_{+|x_n=0^+}|_{\mathcal{H}^{\frac12}}\\ &\qquad\quad + \tau^{-\frac12}\Big(\|\Xi_1 v\|_{L^2(\mathbb{R};\mathcal{H}^2)} + \|H_{-}\Xi_1 D_n v_{-}\|_{L^2(\mathbb{R};\mathcal{H}^1)} + \|H_{+}\Xi_1 D_n v_{+}\|_{L^2(\mathbb{R};\mathcal{H}^1)}\Big),\end{aligned} \tag{4.18}$$

From the estimates (4.17) and (4.18) obtained in each microlocal region, the final derivation of the Carleman estimate of Theorem 1.1 is classical. We assumed the compact set K small here. In fact, this can be relaxed as local Carleman estimates of the type we prove here can be patched together.

5. Necessity of the geometric assumption on the weight function

Considering the operator $\mathcal{L}_\tau$ given by (1.21), we may wonder about the relevance of conditions (1.28) to derive a Carleman estimate. In the simple model and weight used in Section 1.4, it turns out that we can show that condition (1.28) is necessary for an estimate to hold. For simplicity, we consider a *piecewise constant* case $c = H_{+}c_{+} + H_{-}c_{-}$ as in Section 1.4 and we use a weight function of the form of (1.18) here.

Theorem 5.1. *Let us assume that* (1.29) *is violated, i.e.,*

$$\exists \xi'_0 \in \mathbb{R}^{n-1} \setminus 0, \quad \frac{\alpha_+}{\alpha_-} < \frac{m_+(\xi'_0)}{m_-(\xi'_0)}. \tag{5.1}$$

Then, for any neighborhood V of the origin, and $\tau_0 > 0$, there exist

$$v_\tau = H_+ v_{\tau,+} + H_- v_{\tau,-}, \quad \tau \geq \tau_0, \quad v_{\tau,\pm} \in \mathscr{C}_c^\infty(\mathbb{R}^n), \quad \|v_\tau\|_{L^2(\mathbb{R}^n)} = 1,$$

satisfying the transmission conditions (1.19)–(1.20) *at $x_n = 0$, such that*

$$\operatorname{supp}(v_\tau) \subset V \quad \textit{and} \quad \|\mathcal{L}_\tau v\|_{L^2(\mathbb{R}^n)} \underset{\tau\to\infty}{\longrightarrow} 0.$$

The existence of such a quasi-mode v obviously ruins any hope to obtain a Carleman estimate for the operator $\mathcal{L}$ with a weight function satisfying (5.1). The remainder of this section is devoted to this construction.

We set

$$\begin{aligned}(\mathcal{M}_\tau u)(\xi', x_n) = H_+(x_n) c_n^+ \Big(D_n + i e_+\Big)\Big(D_n + i f_+\Big) u_+ \\ + H_-(x_n) c_n^- \Big(D_n + i e_-\Big)\Big(D_n + i f_-\Big) u_-,\end{aligned} \tag{5.2}$$

that is, the action of the operator $\mathcal{L}_\tau$ given in (1.21) in the Fourier domain with respect to x'. We start by constructing a quasi-mode for $\mathcal{M}_\tau$, i.e., functions $u_\pm(\xi', x_n)$ compactly supported in the x_n variable and in a conic neighborhood of ξ'_0 in the variable ξ' with $\|\mathcal{M}_\tau u\|_{L^2} \ll \|u\|_{L^2}$, so that u is nearly an eigenvector of $\mathcal{M}_\tau$ for the eigenvalue 0.

Condition 5.1 implies that there exists $\tau_0 > 0$ such that

$$\frac{m_-(\xi'_0)}{\alpha_-} < \tau_0 < \frac{m_+(\xi'_0)}{\alpha_+}, \quad \text{which gives} \quad \tau_0\alpha_+ - m_+(\xi'_0) < 0 < \tau_0\alpha_- - m_-(\xi'_0).$$

By homogeneity we may in fact choose (τ_0, ξ'_0) such that $\tau_0^2 + |\xi'_0|^2 = 1$. We have thus, using the notation in (1.21),

$$f_+(x_n = 0) = \tau\alpha_+ - m_+(\xi') < 0 < f_-(x_n = 0) = \tau\alpha_- - m_-(\xi'),$$

for (τ, ξ') in a conic neighborhood Γ of (τ_0, ξ'_0) in $\mathbb{R} \times \mathbb{R}^{n-1}$. Let $\chi_1 \in \mathscr{C}_c^\infty(\mathbb{R})$, $0 \leq \chi_1 \leq 1$, with $\chi_1 \equiv 1$ in a neighborhood of 0, such that $\operatorname{supp}(\psi) \subset \Gamma$ with

$$\psi(\tau, \xi') = \chi_1\Big(\frac{\tau}{(\tau^2 + |\xi'|^2)^{\frac{1}{2}}} - \tau_0\Big)\chi_1\Big(\Big|\frac{\xi}{(\tau^2 + |\xi'|^2)^{\frac{1}{2}}} - \xi_0\Big|\Big).$$

We thus have

$$f_+(x_n = 0) \leq -C\tau, \quad C'\tau \leq f_-(x_n = 0) \quad \text{in } \operatorname{supp}(\psi).$$

Let $(\tau, \xi') \in \operatorname{supp}(\psi)$. We can solve the equations

$$\begin{aligned}
&\Big(D_n + i f_+(x_n, \xi'))\Big) q_+ = 0 \quad \text{on } \mathbb{R}_+, \quad f_+(x_n, \xi') = \tau\varphi'(x_n) - m_+(\xi') = f_+(0) + \tau\beta x_n, \\
&\Big(D_n + i f_-(x_n, \xi'))\Big) q_- = 0 \quad \text{on } \mathbb{R}_-, \quad f_-(x_n, \xi') = \tau\varphi'(x_n) - m_-(\xi') = f_-(0) + \tau\beta x_n, \\
&\Big(D_n + i e_-(x_n, \xi'))\Big) \tilde{q}_- = 0 \quad \text{on } \mathbb{R}_-, \quad e_-(x_n, \xi') = \tau\varphi'(x_n) + m_-(\xi') = e_-(0) + \tau\beta x_n,
\end{aligned}$$

that is

$$q_+(\xi', x_n) = Q_+(\xi', x_n) q_+(\xi', 0), \quad Q_+(\xi', x_n) = e^{x_n\left(f_+(0) + \frac{\tau\beta x_n}{2}\right)},$$
$$q_-(\xi', x_n) = Q_-(\xi', x_n) q_-(\xi', 0), \quad Q_-(\xi', x_n) = e^{x_n\left(f_-(0) + \frac{\tau\beta x_n}{2}\right)},$$
$$\tilde{q}_-(\xi', x_n) = \tilde{Q}_-(\xi', x_n) \tilde{q}_-(\xi', 0), \quad \tilde{Q}_-(\xi', x_n) = e^{x_n\left(e_-(0) + \frac{\tau\beta x_n}{2}\right)}.$$

Since $f_+(0) < 0$ a solution of the form of q_+ is a good idea on $x_n \geq 0$ as long as $\tau\beta x_n + 2f_+(0) \leq 0$, i.e., $x_n \leq 2|f_+(0)|/\tau\beta$. Similarly as $f_-(0) > 0$ (resp. $e_-(0) > 0$) a solution of the form of q_- (resp. $\tilde{q}_-$) is a good idea on $x_n \leq 0$ as long as $\tau\beta x_n + 2f_-(0) \geq 0$ (resp. $\tau\beta x_n + 2e_-(0) \geq 0$). To secure this we introduce a cut-off function $\chi_0 \in \mathscr{C}_c^\infty((-1,1);[0,1])$, equal to 1 on $[-\frac{1}{2}, \frac{1}{2}]$ and for $\gamma \geq 1$ we define

$$u_+(\xi', x_n) = Q_+(\xi', x_n)\psi(\tau, \xi')\chi_0\Big(\frac{\tau\beta\gamma x_n}{|f_+(0)|}\Big),$$
$$u_-(\xi', x_n) = aQ_-(\xi', x_n)\psi(\tau, \xi')\chi_0\Big(\frac{\tau\beta\gamma x_n}{f_-(0)}\Big) + b\tilde{Q}_-(\xi', x_n)\psi(\tau, \xi')\chi_0\Big(\frac{\tau\beta\gamma x_n}{e_-(0)}\Big),$$

with $a, b \in \mathbb{R}$, and

$$u(\xi', x_n) = H_+(x_n)u_+(\xi', x_n) + H_-(x_n)u_-(\xi', x_n)$$

The factor γ is introduced to control the size of the support in the x_n direction. Observe that we can satisfy the transmission condition (1.19)–(1.20) by choosing the coefficients a and b. Transmission condition (1.19) implies

$$a + b = 1. \tag{5.3}$$

Transmission condition (1.20) and the equations satisfied by Q_+, Q_- and $\tilde{Q}_-$ imply

$$c_+m_+ = c_-(a-b)m_-. \tag{5.4}$$

We have the following lemma.

Lemma 5.2. *For τ sufficiently large we have*

$$\|\mathcal{M}_\tau u\|^2_{L^2(\mathbb{R}^{n-1}\times\mathbb{R})} \leq C(\gamma^2 + \tau^2)\gamma\tau^{n-1}e^{-C'\tau/\gamma}$$

and

$$\|u\|^2_{L^2(\mathbb{R}^{n-1}\times\mathbb{R})} \geq C\tau^{n-2}\Big(1 - e^{-C'\tau/\gamma}\Big).$$

We now introduce

$$v_\pm(x', x_n) = (2\pi)^{-(n-1)}\chi_0(|\tau^{\frac{1}{2}}x'|)\check{\hat{u}}_\pm(x', x_n) = (2\pi)^{-(n-1)}\chi_0(|\tau^{\frac{1}{2}}x'|)\hat{u}_\pm(-x', x_n),$$

that is, a localized version of the inverse Fourier transform (in x') of $u_\pm$. The functions $v_\pm$ are smooth and compactly supported in $\mathbb{R}^{n-1}_\pm \times \mathbb{R}$ and they satisfy transmission conditions (1.19)–(1.20). We set $v(x', x_n) = H_+(x_n)v_+(x', x_n) + H_-(x_n)v_-(x', x_n)$. In fact we have the following estimates.

Lemma 5.3. *Let $N \in \mathbb{N}$. For τ sufficiently large we have*

$$\|\mathcal{L}_\tau v\|^2_{L^2(\mathbb{R}^{n-1}\times\mathbb{R})} \leq C(\gamma^2 + \tau^2)\gamma\tau^{n-1}e^{-C'\tau/\gamma} + C_{\gamma,N}\tau^{-N}$$

and

$$\|v\|^2_{L^2(\mathbb{R}^{n-1}\times\mathbb{R})} \geq C\tau^{n-2}\Big(1 - e^{-C'\tau/\gamma}\Big) - C_{\gamma,N}\tau^{-N}.$$

We may now conclude the proof of Theorem 5.1. In fact, if V is an arbitrary neighborhood of the origin, we choose τ and γ sufficiently large so that $\operatorname{supp}(v) \subset V$. We then keep γ fixed. The estimates of Lemma 5.3 show that

$$\|\mathcal{L}_\tau v\|_{L^2(\mathbb{R}^{n-1}\times\mathbb{R})} \|v\|^{-1}_{L^2(\mathbb{R}^{n-1}\times\mathbb{R})} \underset{\tau\to\infty}{\longrightarrow} 0.$$

Remark 5.4. As opposed to the analogy we give at the beginning of Section 1.5, the construction of this quasi-mode does not simply rely on one of the first-order factor. The transmission conditions are responsible for this fact. In very particular cases it could only involve the factors $D_n + if_\pm$ on both sides. In the more general case, as studied in this section, the construction relies on the factor $D_n + if_+$ in $x_n \geq 0$, i.e., a one-dimensional space of solutions, and on both factors $D_n + if_-$ and $D_n + ie_-$ in $x_n \geq 0$, i.e., a two-dimensional space of solutions.

Appendix A. A few facts on pseudo-differential operators

A.1. Standard classes and Weyl quantization

We define for $m \in \mathbb{R}$ the class of symbols $\mathcal{S}^m$ as the smooth functions on $\mathbb{R}^n \times \mathbb{R}^{n-1}$ such that, for all $(\alpha, \beta) \in \mathbb{N}^n \times \mathbb{N}^{n-1}$,

$$N_{\alpha\beta}(a) = \sup_{(x,\xi')\in\mathbb{R}^n\times\mathbb{R}^{n-1}} \langle\xi'\rangle^{-m+|\beta|} |(\partial_x^\alpha \partial_{\xi'}^\beta a)(x,\xi')| < \infty, \tag{A.1}$$

with $\langle\xi'\rangle^2 = 1 + |\xi'|^2$. The quantities on the l.h.s. above are called the semi-norms of the symbol a. For $a \in \mathcal{S}^m$, we define $\operatorname{op}(a)$ as the operator defined on $\mathscr{S}(\mathbb{R}^n)$ by

$$(\operatorname{op}(a)u)(x', x_n) = a(x, D')u(x', x_n) = \int_{\mathbb{R}^{n-1}} e^{ix'\cdot\xi'} a(x', x_n, \xi')\hat{u}(\xi', x_n) d\xi' (2\pi)^{1-n}, \tag{A.2}$$

with $(x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R}$, where $\hat{u}$ is the partial Fourier transform of u with respect to the variable x'. For all $(k, s) \in \mathbb{Z} \times \mathbb{R}$ we have

$$\operatorname{op}(a) : H^k(\mathbb{R}_{x_n}; H^{s+m}(\mathbb{R}^{n-1}_{x'})) \to H^k(\mathbb{R}_{x_n}; H^s(\mathbb{R}^{n-1}_{x'})) \quad \text{continuously}, \tag{A.3}$$

and the norm of this mapping depends only on $\{N_{\alpha\beta}(a)\}_{|\alpha|+|\beta|\leq\mu(k,s,m,n)}$, where $\mu : \mathbb{Z} \times \mathbb{R} \times \mathbb{R} \times \mathbb{N} \to \mathbb{N}$.

We shall also use the *Weyl quantization* of a denoted by $\operatorname{op}^w(a)$ and given by the formula

$$\begin{aligned}(\operatorname{op}^w(a)u)(x', x_n) &= a^w(x, D')u(x', x_n) \\ &= \iint_{\mathbb{R}^{2n-2}} e^{i(x'-y')\cdot\xi'} a\Big(\frac{x'+y'}{2}, x_n, \xi'\Big) u(y', x_n) dy' d\xi' (2\pi)^{1-n}.\end{aligned} \tag{A.4}$$

Property (A.3) holds as well for $\operatorname{op}^w(a)$. A nice feature of the Weyl quantization that we use in this article is the simple relationship with adjoint operators with the formula

$$\Big(\operatorname{op}^w(a)\Big)^* = \operatorname{op}^w(\bar{a}), \tag{A.5}$$

so that for a real-valued symbol $a \in \mathcal{S}^m$ $(\mathrm{op}^w(a))^* = \mathrm{op}^w(a)$. We have also for $a_j \in \mathcal{S}^{m_j}$, $j = 1, 2$,

$$\mathrm{op}^w(a_1)\mathrm{op}^w(a_2) = \mathrm{op}^w(a_1 \sharp a_2), \quad a_1 \sharp a_2 \in \mathcal{S}^{m_1+m_2}, \tag{A.6}$$

$$\mathrm{op}^w(a_1)\mathrm{op}^w(a_2) = \mathrm{op}^w(a_1 a_2) + \mathrm{op}^w(r_1), \quad r_1 \in \mathcal{S}^{m_1+m_2-1}, \tag{A.7}$$

$$\text{with } r_1 = \frac{1}{2i}\{a_1, a_2\} + r_2, \quad r_2 \in \mathcal{S}^{m_1+m_2-2}, \tag{A.8}$$

$$[\mathrm{op}^w(a_1), \mathrm{op}^w(a_2)] = \mathrm{op}^w(\frac{1}{i}\{a_1, a_2\}) + \mathrm{op}^w(r_3), \quad r_3 \in \mathcal{S}^{m_1+m_2-3}, \tag{A.9}$$

where $\{a_1, a_2\}$ is the Poisson bracket. Moreover, for $b_j \in \mathcal{S}^{m_j}$, $j = 1, 2$, both real-valued, we have

$$[\mathrm{op}^w(b_1), i\mathrm{op}^w(b_2)] = \mathrm{op}^w(\{b_1, b_2\}) + \mathrm{op}^w(s_3), \quad s_3 \text{ real-valued} \in \mathcal{S}^{m_1+m_2-3}. \tag{A.10}$$

Lemma A.1. *Let $a \in \mathcal{S}^1$ such that $a(x, \xi') \geq \mu\langle\xi'\rangle$, with $\mu \geq 0$. Then there exists $C > 0$ such that*

$$\mathrm{op}^w(a) + C \geq \mu\langle D'\rangle, \qquad (\mathrm{op}^w(a))^2 + C \geq \mu^2\langle D'\rangle^2.$$

A.2. Pseudo-differential calculus with a large parameter

We let $\tau \in \mathbb{R}$ be such that $\tau \geq \tau_0 \geq 1$. We set $\lambda^2 = 1 + \tau^2 + |\xi'|^2$. We define for $m \in \mathbb{R}$ the class of symbols $\mathcal{S}(\lambda^m)$ as the smooth functions on $\mathbb{R}^n \times \mathbb{R}^{n-1}$, depending on the parameter τ such that, for all $(\alpha, \beta) \in \mathbb{N}^n \times \mathbb{N}^{n-1}$,

$$N_{\alpha\beta}(a) = \sup_{\substack{(x,\xi')\in\mathbb{R}^n\times\mathbb{R}^{n-1}\\ \tau\geq\tau_0}} \lambda^{-m+|\beta|}|(\partial_x^\alpha \partial_{\xi'}^\beta a)(x, \xi', \tau)| < \infty. \tag{A.11}$$

The associated operators are defined by (A.2). We can introduce Sobolev spaces and Sobolev norms which are adapted to the scaling large parameter τ. Let $s \in \mathbb{R}$; we set

$$\|u\|_{\mathcal{H}^s} := \|\Lambda^s u\|_{L^2(\mathbb{R}^{n-1})}, \quad \text{with } \Lambda^s := \mathrm{op}(\lambda^s)$$

and

$$\mathcal{H}^s = \mathcal{H}^s(\mathbb{R}^{n-1}) := \{u \in \mathcal{S}'(\mathbb{R}^{n-1});\ \|u\|_{\mathcal{H}^s} < \infty\}.$$

The space $\mathcal{H}^s$ is algebraically equal to the classical Sobolev space $H^s(\mathbb{R}^{n-1})$, which norm is denoted by $\|.\|_{H^s}$. For $s \geq 0$ note that we have

$$\|u\|_{\mathcal{H}^s} \sim \tau^s\|u\|_{L^2(\mathbb{R}^{n-1})} + \|\langle D'\rangle^s u\|_{L^2(\mathbb{R}^{n-1})}.$$

If $a \in \mathcal{S}(\lambda^m)$ then, for all $(k, s) \in \mathbb{Z} \times \mathbb{R}$, we have

$$\mathrm{op}(a) : H^k(\mathbb{R}_{x_n}; \mathcal{H}^{s+m}) \to H^k(\mathbb{R}_{x_n}; \mathcal{H}^s(\mathbb{R}_{x'}^{n-1})) \quad \text{continuously,} \tag{A.12}$$

and the norm of this mapping depends only on $\{N_{\alpha\beta}(a)\}_{|\alpha|+|\beta|\leq\mu(k,s,m,n)}$, where $\mu : \mathbb{Z} \times \mathbb{R} \times \mathbb{R} \times \mathbb{N} \to \mathbb{N}$.

For the calculus with a large parameter we shall also use the Weyl quantization of (A.4). All the formulæ listed in (A.5)–(A.10) hold as well, with $\mathcal{S}^m$ everywhere replaced by $\mathcal{S}(\lambda^m)$. We use the Gårding inequality as stated in the following lemma.

Lemma A.2. *Let $a \in \mathcal{S}(\lambda^m)$ such that $\mathrm{Re}\, a \geq C\lambda^m$. Then*

$$\mathrm{Re}(\mathrm{op}^w(a)u, u) \gtrsim \|u\|^2_{L^2(\mathbb{R};\mathcal{H}^{\frac{m}{2}})},$$

for τ sufficiently large.

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