*Journées*

**ÉQUATIONS AUX DÉRIVÉES PARTIELLES**

Obernai, 11–15 juin 2018

Evelyne Miot

**On some coupled PDE-ODE systems in fluid dynamics**

*J. É. D. P.* (2018), Exposé n$^{o}$ V, 13 p.

<http://jedp.cedram.org/item?id=JEDP_2018____A5_0>

cedram

*Article mis en ligne dans le cadre du*
*Centre de diffusion des revues académiques de mathématiques*
http://www.cedram.org/

GROUPEMENT DE RECHERCHE 2434 DU CNRS

Journées *Équations aux dérivées partielles*
Obernai, 11–15 juin 2018
GDR 2434 (CNRS)

# On some coupled PDE-ODE systems in fluid dynamics

Evelyne Miot

**Abstract**

In this note we will present some existence and uniqueness issues for three coupled PDE-ODE systems. The common frame is that they arise as the asymptotical dynamics of a regular, incompressible two-dimensional flow interacting with:

- points at which the vorticity is highly concentrated (point vortices);
- an obstacle shrinking to a steady point;
- rigid bodies contracting to moving massive particles.

We will mainly focus on the last situation, corresponding to the article [11], which is a joint work with Christophe Lacave.

## 1. Introduction

### 1.1. Setting

The purpose of this note is to study the evolution of a two-dimensional incompressible flow interacting with one or several point singularities, in three different settings. The results presented here correspond to the papers [10] and mostly [11].

Given a two-dimensional, incompressible inviscid fluid, we consider the (divergence-free) velocity of the fluid: $u = u(t,x) : \mathbb{R}_+ \times \mathbb{R}^2 \to \mathbb{R}^2$ and the vorticity $\omega = \mathrm{curl}(u) : \mathbb{R}_+ \times \mathbb{R}^2 \to \mathbb{R}$.

In the absence of point singularities, the evolution of the velocity is given by the Euler equation

$$\partial_t u + u \cdot \nabla u = -\nabla p, \quad \mathrm{div}\, u = 0, \tag{1.1}$$

or, when expressed in terms of the vorticity,

$$\partial_t \omega + u \cdot \nabla \omega = 0, \quad \mathrm{div}\, u = 0. \tag{1.2}$$

Note that, under decay conditions at infinity, $u$ can be explicitely recovered in terms of $\omega$ by the Biot–Savart law : $u = K * \omega$, with $K(x) = x^\perp/(2\pi|x|^2)$. Global existence and uniqueness of classical solutions to (1.2) was proved in [8, 21]. Yudovich [22] established global existence and uniqueness of the weak solution $\omega \in L^\infty(\mathbb{R}_+, L^1 \cap L^\infty(\mathbb{R}^2))$, for all initial vorticity belonging to $L^1 \cap L^\infty(\mathbb{R}^2)$. As in the classical case, the weak solution is also a *lagrangian* solution, namely it is transported by the unique lagrangian flow of the velocity field. We refer e.g. to the book of Majda & Bertozzi [12] for a more detailed presentation of the Euler equation.

In this article, we will review three situations in which a regular fluid, with uniformly bounded and integrable vorticity $\omega(t,\cdot)$, interacts with one or several point singularities located at $z_1(t), \dots, z_N(t)$ at time $t \geq 0$. Then (1.2) has to be modified according to the singular field generated by the singularities. This yields a coupled PDE/ODE system for which we will study the main properties: existence, uniqueness and lagrangian representation of the solution $\omega(t,\cdot)$. We describe below the situations under consideration.

E. M. is supported by the French ANR projects SchEq ANR-12-JS-0005-01, GEODISP ANR-12-BS01-0015-01, and INFAMIE ANR-15-CE40-01.

1. *Point vortices.* The point singularities correspond to points at which part of the vorticity is highly concentrated: it behaves as a sum of Dirac masses $\sum_k \gamma_k \delta_{z_k(t)}$, where $\gamma_k$ is the circulation of the velocity around $z_k(t)$. In this setting the points are called *point vortices.* The corresponding interaction is given by the vortex-wave system:
$$\begin{cases} \partial_t \omega + \operatorname{div}\left[\left(u + \sum_{k=1}^N \frac{\gamma_k}{2\pi}\frac{(x-z_k)^\perp}{|x-z_k|^2}\right)\omega\right] = 0, \\ u = K * \omega, \quad K(x) = \frac{1}{2\pi}\frac{x^\perp}{|x|^2}, \\ \dot{z}_k = u(t,z_k) + \sum_{j\neq k} \frac{\gamma_j}{2\pi}\frac{(z_k-z_j)^\perp}{|z_k-z_j|^2} \quad \text{for } k=1,\dots,N. \end{cases} \tag{1.3}$$
This is a coupling of a PDE for the motion of the regular flow, with vorticity $\omega(t,\cdot)$ uniformly integrable and bounded as in Yudovich's theorem, and a system of ODEs for the point vortices. The vortex-wave system (1.3) was introduced and studied by Marchioro and Pulvirenti [14, 15] and Starovoĭtov [18]. Marchioro and Pulvirenti proved global existence of a (lagrangian) solution when $\gamma_k$ all have the same sign. The sign condition prevents from collisions in finite time. In [14], they also indicated that uniqueness holds when the vorticity is initially constant near the point vortices (namely the condition appearing in Theorem 1.3 below). This was proved in [10], as stated in Theorem 1.3 below. We also quote [19] for a uniqueness statement with an additional regularity condition.

Note that the vortex-wave system reduces to the point vortex system (or Kirschoff law) when there is no regular fluid,
$$\dot{z}_k = \sum_{j\neq k} \frac{\gamma_j}{2\pi}\frac{(z_k-z_j)^\perp}{|z_k-z_j|^2} \quad \text{for } k=1,\dots,N.$$

2. *Fixed point vortex.* There is one single point ($N=1$), which is stationnary. This corresponds to the asymptotics of a regular fluid evolving in the exterior of a small, compact obstacle shrinking into a fixed point $z_1(t)\equiv z_1$, in a self-similar way, with constant circulation around the obstacle. Assuming that $z_1$ is set at the origin, the asymptotical dynamics is given by
$$\begin{cases} \partial_t \omega + \operatorname{div}\left[\left(u + \frac{\gamma}{2\pi}\frac{x^\perp}{|x|^2}\right)\omega\right] = 0 \\ u = K*\omega, \end{cases} \tag{1.4}$$
where $\gamma$ is reminiscent of the circulation of the velocity field around the obstacle. System (1.4) was derived by Iftimie, Lopes Filho and Nussenzveig Lopes [9]. On the other hand, it had been previously studied by Marchioro [13], who established existence and uniqueness of the classical solution with compact support not intersecting the origin. See also [10] (or Theorem 1.3 below) for an alternative proof of uniqueness.

3. *Massive point vortices.* The points correspond to the asymptotical positions of small, rigid bodies immerged in the regular fluid, when the size of the bodies vanishes with fixed mass $m_k > 0$ and circulation $\gamma_k \in \mathbb{R}$. The resulting system reads:
$$\begin{cases} \partial_t \omega + \operatorname{div}\left[\left(u + \sum_{k=1}^N \frac{\gamma_k}{2\pi}\frac{(x-z_k)^\perp}{|x-z_k|^2}\right)\omega\right] = 0, \\ u = K*\omega, \\ m_k \ddot{z}_k = \gamma_k \left(\dot{z}_k - u(t,z_k) - \sum_{j\neq k}\frac{\gamma_j}{2\pi}\frac{(z_k-z_j)^\perp}{|z_k-z_j|^2}\right)^\perp \\ \text{ for } k=1,\dots,N. \end{cases} \tag{1.5}$$
We observe that (1.5) reduces to the vortex-wave system (1.3) when setting $m_k = 0$. And, for $N=1$, Glass, Lacave and Sueur [7] proved that the asymptotical dynamics of a small body with vanishing mass evolving in a 2D incompressible fluid is indeed governed by the vortex-wave system.

The second order differential equation satisfied by the point vortices in (1.5) means that the bodies are accelerated by a force, which is analogous to the well-known Kutta–Joukowski-type lift force that arises for a single body in an irrotational unbounded flow. The properties of System (1.5) were investigated in [6, 7]. In the case of one single point $N=1$, the corresponding system was rigorously derived by Glass, Lacave and Sueur [6] in a distinguished limit when the size of the body vanishes whereas the mass is assumed to be constant. Thus, a byproduct of [6] is the existence of a global weak solution of (1.5) when $N = 1$.

In the case $N \geq 1$, the derivation of (1.5) is an open issue. One of the results in the present note, given in Theorem 1.1 below, provides existence and (in some cases) uniqueness of the solution for any $N \geq 1$, that is global if all the circulations have the same sign. In particular, this could be a first step to rigorously justify the mean-field limit of (1.5) when setting $\gamma_k = 1/N = m_k$ and letting $N \to +\infty$. Formally, this mean-field limit is a system of coupled PDE for the couple $(\omega, f)$, where

$$f = f(t, x, v)$$

is the weak-$*$ limit of the empirical measure

$$f_N(t) = \frac{1}{N} \sum_{k=1}^{N} \delta_{(z_k(t), \dot{z}_k(t))}$$

as $N \to +\infty$. The system reads

$$\begin{cases} \partial_t \omega + \operatorname{div}\left[(u + K * \rho)\, \omega\right] = 0, \\ u = K * \omega, \quad K(x) = \frac{1}{2\pi} \frac{x^\perp}{|x|^2}, \\ \partial_t f + v \cdot \nabla_x f + (v^\perp - u^\perp) \cdot \nabla_v f = 0, \end{cases} \tag{1.6}$$

with

$$\rho = \rho(t, x) = \int f(t, x, v)\, \mathrm{d}v.$$

System (1.6) is investigated by Moussa and Sueur [17] as a model for the evolution of 2D sprays. In this case, $f$ denotes the density of a dispersed phase of particles moving into a perfect fluid with velocity $u$. The mean-field limit of (1.5) to (1.6) is proved in [17] in the regularized case, that is when $K$ is replaced by a Lipschhitz and bounded field. The "real" case $K(x) = x^\perp/(2\pi|x|^2)$ is open.

### 1.2. Main results

We state here the main results holding for the aforementioned situations. In all the following we consider an initial vorticity

$$\omega_0 \in L^\infty(\mathbb{R}^2), \quad \text{such that} \quad \operatorname{supp}(\omega_0) \subset B(0, R_0).$$

We consider distinct points $z_1^0, \ldots, z_N^0$ in $\mathbb{R}^2$ as well as points $h_1, \ldots, h_N^0$ in $\mathbb{R}^2$. The points $z_k^0$ correspond to the initial positions of the point vortices for (1.3) or for (1.5), while the $h_k^0$ are the initial velocities for (1.5). For the fixed point vortex system these initial data will of course not be needed.

**Theorem 1.1.** *We assume that all the $\gamma_k$ have the same sign. Then for* (1.3), (1.4), *and* (1.5)*:*

- *There exists a global weak solution with $\omega \in L^\infty(\mathbb{R}_+, L^1 \cap L^\infty(\mathbb{R}^2))$ and $z_k \in C^1(\mathbb{R}_+)$ for* (1.3), *$z_k \in C^2(\mathbb{R}_+)$ for* (1.5).
- *No collision occurs between the point vortices.*
- *$\|\omega(t, \cdot)\|_{L^p} = \|\omega_0\|_{L^p}$ for all $1 \leq p \leq +\infty$ and for all $t \geq 0$.*
- *The solution is a lagrangian solution: we have $\omega(t, \cdot) = X(t, \cdot)_\# \omega_0$, where for almost every $x \neq z_k^0$,*

$$\begin{cases} \dot{X}(t, x) = u(t, X(t, x)) + \sum_{k=1}^{N} \gamma_k K(X(t, x) - z_k(t)), \\ X(0, x) = x. \end{cases} \tag{1.7}$$

*In particular, we have $X(t, x) \neq z_k(t)$ for all $t \in \mathbb{R}_+$.*

**Remark 1.2.** If all the $\gamma_k$ do not have the same sign, the previous results apply on some time interval $[0, T]$ with $T > 0$.

As mentioned above, Theorem 1.1 was proved in [14] for (1.3), in [10, 13] for (1.4), and in [11] for (1.5).

Concerning uniqueness, our result is the following:

**Theorem 1.3.** *Assume that there exist* $\alpha_1, \dots, \alpha_N \in \mathbb{R}$ *and some* $\delta_0 > 0$ *such that*

$$\omega_0 \equiv \alpha_k \quad \textit{on} \quad B(z_k^0, \delta_0), \quad k = 1, \dots, N.$$

*Then for any* $T > 0$*, there exists at most one weak solution* $(\omega, \{z_k\})$ *to* (1.3)*,* (1.4) *or* (1.5) *on* $[0, T]$ *with this initial condition.*

As already mentioned, Theorem 1.3 was sketched in [14] and proved in [10] for the vortex-wave system (1.3). It was proved in [10, 13] for the system (1.4) when $\alpha_1 = 0$ (the vorticity vanishes near the point vortex). Finally it is established in [11] for the massive vortex-wave system (1.5).

The strategy for proving Theorem 1.3, which was indicated by Marchioro and Pulvirenti for the vortex-wave system, relies on the property that for all $t \in [0, T]$, the vorticity remains constant in the neighborhood of the massive point vortices:

**Theorem 1.4.** *Let* $\omega_0$ *and* $\{z_{0,k}\}$ *satisfy the assumptions of Theorem 1.3. Let* $(\omega, \{z_k\})$ *be any weak solution of* (1.3)*,* (1.4) *or* (1.5) *on* $[0, T]$*. There exists a positive* $\delta$ *depending only on* $T$*,* $\delta_0$*,* $\|\omega_0\|_{L^\infty}$*,* $R_0$ *and* $\|h_k\|_{W^{2,\infty}([0,T])}$*, such that*

$$\omega(t, \cdot) = \alpha_k \quad \textit{on } B(h_k(t), \delta), \quad \forall\, t \in [0, T].$$

As we shall see in the proofs, Theorems 1.3 and 1.4 actually apply to any system of the form

$$\begin{cases} \partial_t \omega + \operatorname{div}\left[\left(u + \sum_{k=1}^N \frac{\gamma_k}{2\pi} \frac{(x - z_k)^\perp}{|x - z_k|^2}\right) \omega\right] = 0, \\ u = K * \omega, \\ z_k \text{ is a given trajectory belonging to } W^{2,\infty}([0, T], \mathbb{R}^2) \text{ for } k = 1, \dots, N. \end{cases}$$

Indeed, the derivative of the local energy defined in (2.10), used to control the distances between the trajectories, only involves estimates on the second-order derivatives of the point trajectories, but does not involve their explicit dynamics.

*From now on, to simplify the presentation, we will focus on the system* (1.5) *with massive point vortices.*

The plan of this paper is as follows. In the next section we sketch the proof of Theorem 1.1 for (1.5). In particular we recall the Definition 2.2 of the regular Lagrangian flow $X$, which is defined almost-everywhere. Then we introduce in (2.10) the notion of *local energy* associated to the flow trajectories $X(t, x)$. By controlling this energy we estimate from below the distance between the flow trajectories and the point vortices globally in time.

Then in Section 3 we sketch the proof of Theorem 1.3 for (1.5). The first step consists in using the previous control on the local energies to establish Theorem 1.4. The uniqueness follows then from Theorem 1.4, by mimicking the proof of the paper [10] for the vortex-wave system.

**Notations.** From now on $C$ will refer to a constant depending only on the initial data: $R_0$, $m_k$, $\gamma_k$, and $\|\omega_0\|_{L^\infty}$, but not on $\delta_0$.

For $T > 0$, the notation $C_{\|z\|,T}$ will stand for a constant depending only on the initial data: $R_0$, $m_k$, $\gamma_k$, $\|\omega_0\|_{L^\infty}$, and on $T$ and on $\|z_k\|_{W^{2,\infty}([0,T])}$, but not on $\delta_0$.

For $T > 0$, the notation $C_T$ will refer to a constant depending only on $T$ and on the initial data ($R_0$, $m_k$, $\gamma_k$, and $\|\omega_0\|_{L^\infty}$), but not on $\delta_0$.

$C$, $C_{\|z\|,T}$ and $C_T$ will possibly change value from one line to another.

## 2. Proof of Theorem 1.1 for (1.5)

In all this section we fix $T > 0$ and we perform the estimates on $[0, T]$ under the assumptions of Theorem 1.1.

## 2.1. Global existence of a weak solution

We show the first two items of Theorem 1.1. The method is classical: construction of an approximate weak solution via an iterative scheme, uniform bounds, and passing to the limit. All subsequent proofs are sketched.

### Step 1: iterative scheme and uniform bounds

We consider the following scheme: for $n \in \mathbb{N}^*$ and given

$$\omega_{n-1} \in L^\infty([0,T], L^1 \cap L^\infty(\mathbb{R}^2)),$$

we set

$$u_{n-1} = K * \omega_{n-1}.$$

Our purpose is to solve the linear PDE

$$\begin{cases} \partial_t \omega_n + \left( u_{n-1} + \sum_{k=1}^N \frac{\gamma_k}{2\pi} \frac{(x - z_{k,n-1})^\perp}{|x - z_{j,n-1}|^2} \right) \cdot \nabla \omega_n = 0 \\ \omega_n(0, \cdot\,) = \omega_0, \end{cases} \tag{2.1}$$

and the linear system of ODEs: for $k = 1, \dots, N$,

$$\begin{cases} m_k \ddot{z}_{k,n} = \gamma_k \left( \dot{z}_{k,n} - u_{n-1}(t, z_{k,n}) - \sum_{j \neq k} \frac{\gamma_j}{2\pi} \frac{(z_{j,n} - z_{k,n})^\perp}{|z_{j,n} - z_{k,n}|^2} \right)^\perp \\ (z_{k,n}(0), \dot{z}_{k,n}(0)) = (z_k^0, h_k^0). \end{cases} \tag{2.2}$$

For $n = 0$ we take $\omega_0$ and $(z_k^0, h_k^0)$ as data.

The following proposition indeed yields a solution for each $n$:

**Proposition 2.1.** *For all $n \in \mathbb{N}$, there exists a unique weak solution*

$$\omega_n \in L^\infty([0,T], L^1 \cap L^\infty(\mathbb{R}^2))$$

*to* (2.1) *and a unique solution*

$$z_{k,n} \in C^2([0,T]), \quad k = 1, \dots, N$$

*to* (2.2) *on* $[0,T]$. *Moreover*

$$\|\omega_n\|_{L^1 \cap L^\infty} \leq M_T, \quad \min_{j \neq k} \min_{t \in [0,T]} |z_{j,n}(t) - z_{k,n}(t)| \geq d_T,$$

*where $M_T$ and $d_T > 0$ depend only on $T$ and on the initial data.*

*Proof of Proposition 2.1.* We refer to [11] for the detailed proof. In particular, the uniform lower bound on the distances is a consequence of the control of the quantity (related to the hamiltonian of the system, see Proposition 4.2)

$$\mathcal{H}_n(t) = \sum_{j \neq k} \frac{\gamma_j \gamma_k}{2\pi} \ln |z_{j,n}(t) - z_{k,n}(t)| - \sum_{k=1}^N m_k |\dot{z}_{k,n}(t)|^2,$$

which is proved to be uniformly bounded on $[0,T]$. This, together with the sign condition on the circulations, yields the lower bound. □

### Step 2: passing to the limit

The existence of $\omega$ and $z_k, k = 1, \dots, N$ such that (up to a subsequence) $\{\omega_n\}_{n \in \mathbb{N}}$ converges to $\omega$ in $L^\infty$ weak - $*$ and such that each $\{(z_{k,n}, \dot{z}_{k,n})\}_{n \in \mathbb{N}}$ converges uniformly to $(z_k, \dot{z}_k)$, follows from the bounds of Proposition 2.1, Banach–Alaoglu's theorem and Ascoli's theorem. In particular, setting $u = K * \omega$, we infer that $\{u_n = K * \omega_n\}_{n \in \mathbb{N}}$ converges to $u = K * \omega$ locally uniformly on $[0,T] \times \mathbb{R}^2$. So, we can pass to the limit in the iterative scheme and finally show that $(\omega, \{z_k\})$ is a weak solution of (1.5). Note in particular that

$$\max_k \max_{[0,T]} |z_k| \leq C_T, \quad \min_{j \neq k} \min_{[0,T]} |z_j - z_k| = d_T > 0 \tag{2.3}$$

and

$$\sup_k \sup_{[0,T]} |\dot{z}_k| \leq C_T. \tag{2.4}$$

## 2.2. Any weak solution is a lagrangian solution

We now prove the last item of Theorem 1.1. In all this paragraph, $(\omega, \{z_k\})$ denotes any weak solution of (1.5) on $[0,T]$. Since $\omega$ belongs to $L^\infty([0,T], L^1 \cap L^\infty(\mathbb{R}^2))$ it is well-known that $u$ is almost-Lipschitz, see e.g. [15, Appendix 2.3]:

$$\sup_{t\in[0,T]} |u(t,x) - u(t,y)| \leq C|x-y|(1 + |\ln|x-y||), \quad \forall\, (x,y) \in \mathbb{R}^2 \times \mathbb{R}^2. \tag{2.5}$$

We also have the Calderón–Zygmund inequality [20, Chapter II, Theorem 3]: there exists $C > 0$ such that for all $p \geq 2$,

$$\sup_{t\in[0,T]} \|\nabla u(t,\cdot)\|_{L^p} \leq C\,p.$$

In particular, it follows that

$$u \in L^\infty([0,T], L^\infty(\mathbb{R}^2)) \cap L^\infty([0,T], W^{1,1}_{\mathrm{loc}}(\mathbb{R}^2)), \quad \mathrm{div}(u) = 0. \tag{2.6}$$

Finally, $u$ is continuous on $[0,T] \times \mathbb{R}^2$, see [10, Proposition 4.1].

### A general abstract result for linear transport equations

We start by recalling the definition of regular Lagrangian flow, formulated in the papers [1, 2, 3, 4]:

**Definition 2.2.** *Let $T > 0$ and let $b \in L^1_{\mathrm{loc}}([0,T] \times \mathbb{R}^2)$. We say that $X : [0,T] \times \mathbb{R}^2 \times \mathbb{R}^2$ is a regular Lagrangian flow relative to $b$ if*

- *For $\mathcal{L}^2$-a.e. $x \in \mathbb{R}^2$, the map $t \mapsto X(t,x)$ is an absolutely continuous solution to the ODE $\frac{\mathrm{d}}{\mathrm{d}t}X(t,x) = b(t, X(t,x))$ with $X(0,x) = x$;*

- *For all $R > 0$ there exists $L_R > 0$ such that*[1]

$$X(t,\cdot)_\#(\mathcal{L}^d \llcorner B_R) \leq L_R \mathcal{L}^d, \quad \forall\, t \in [0,T].$$

Such a definition is intended to generalize the classical notion of flow associated to smooth vector fields. Existence and uniqueness of the generalized Lagrangian flow[2] were established by DiPerna and Lions [5] when the velocity field has Sobolev-type spacial regularity. This was later extended by Ambrosio [1] to BV vector fields.

In the present setting, we deal with (divergence free) vector fields

$$b(t,x) = u(t,x) + \sum_{k=1}^{N} \gamma_k K(x - z_k(t)), \tag{2.7}$$

that are composed of a "regular" part $u$ (satisfing (2.6)) and of a part with some localized singularities created by the point vortices. So this kind of behavior is not covered by the previous results proved in [5, 1].

Therefore, to deal with such vector fields, we shall use again the strategy applied in [4, 10] for the vortex-wave system (1.3). More precisely, we shall invoke the following abstract result by Ambrosio [1, Theorems 3.3 and 3.5]: let a vector field $b$ in $L^1_{\mathrm{loc}}([0,T] \times \mathbb{R}^2)$, if existence and uniqueness for the continuity equation

$$\partial_t \omega + \mathrm{div}(b\omega) = 0, \quad \omega(0,\cdot) = \omega_0 \in L^1 \cap L^\infty(\mathbb{R}^2)$$

hold in $L^\infty([0,T], L^1 \cap L^\infty(\mathbb{R}^2))$, then the regular Lagrangian flow $X$ for $b$ exists and is unique. Moreover, the unique solution is given by $\omega(t,\cdot) = X(t,\cdot)_\#\omega_0$.

[1]This means that $\mathcal{L}^d(X(t,\cdot)^{-1}(A) \cap B(0,R)) \leq L_R \mathcal{L}^d(A)$, for all Borel set $A$ of $\mathbb{R}^2$.

[2]The definition of flow in [5] slightly differs from the more recent one given in Definition 2.2 but both are essentially equivalent.

Now, it was proved in [16, Chapter 1, Lemme 1.5] [3] and in [10, Lemma 3.2] for the case of one point, that the transport equation associated to the divergence free velocity field $b$ defined in (2.7) admits a unique solution, which is renormalized.[4]

Finally, combining this with Ambrosio's result yields the existence and uniqueness of the regular Lagrangian flow $X$ associated to $b$. Furthermore, the solution is transported by this flow.

On the other hand, as noted in [16, Chapter 1, Remark 1.3] or in [10, Remark 3.3] for the case of one point, the renormalization property ensures that

$$\|\omega(t,\cdot)\|_{L^p} = \|\omega_0\|_{L^p}, \quad 1 \leq p \leq +\infty. \tag{2.8}$$

This shows the third item of Theorem 1.1.

At this stage, in order to show the last item of Theorem 1.1, we still need to show that $X$ satisfies for almost every $x \neq z_k^0$:

$$X(t,x) \neq z_k(t), \quad \forall\, t \in [0,T]. \tag{2.9}$$

In particular, we observe that, in view of the continuity of $u$, this will show that $X(\cdot, x)$ is Lipschitz continuous. The next paragraph is devoted to the proof of (2.9).

## 2.3. Local energy

We introduce the following notion of pointwise energy. For a.e. $x$ such that $X(t,x) \neq z_j(t)$ for any $j = 1, \ldots, N$ on the maximal interval $[0, T(x))$, we set:

$$F_k(t) = \sum_{j=1}^{N} \frac{\gamma_j}{2\pi} \ln|X(t,x) - z_j(t)| + \varphi(t, X(t,x)) + \langle X(t,x), \dot{z}_k^{\perp}(t)\rangle - 1, \tag{2.10}$$

where we define the stream function[5]

$$\varphi(t,x) = \frac{1}{2\pi} \int_{\mathbb{R}^2} \ln|x-y| \omega(t,y)\, \mathrm{d}y, \tag{2.11}$$

so that

$$u(t,x) = \nabla^{\perp}\varphi(t,x).$$

Our next purpose is to obtain a lower bound on $F_k(t)$, which will in turn provide a lower bound for $|X_k(t,x) - z_k(t)|$.

**Remark 2.3.** The analogous quantity to estimate from below the minimal distance between the flow trajectories and the point singularities is defined for the vortex-wave system (1.3) in [14] as

$$F(t) = \sum_{j=1}^{N} |X(t,x) - z_j(t)|^{-2},$$

and for the fixed point vortex (1.4) in [13] as

$$F(t) = \frac{\gamma}{2\pi} \ln|X(t,x)| + \varphi(t, X(t,x)) - 1.$$

**Proposition 2.4.** *We have for $t \in [0, T^*(x))$ and for all $k = 1, \ldots, N$,*

$$|F_k'(t)| \leq C_T\big(|x| + \sum_{j \neq k} |X(t,x) - h_j(t)|^{-1}\big).$$

[3] Although [16, Lemme 1.5] is stated for the vortex-wave system, we remark that this Lemma holds for any linear transport equation with vector field $b$ given by (2.7), where $u$ satisfies the regularity properties (2.6) and where the point trajectories are Lipschitz continuous on $[0,T]$ and do not intersect. Since their precise dynamics is not used to show the renormalization property, the result of [16] does hold in the present case.

[4] This means that for any continuous function $\beta$ growing not too fast at infinity, the function $\beta(\omega)$ is also a solution.

[5] Actually in [11] the local energy is defined in terms of a sequence $\varphi_\varepsilon$ of regularizations of $\varphi$, the estimates are performed uniformly for each $\varepsilon$, and the final estimate is obtained by letting $\varepsilon$ go to zero.

*Proof of Proposition 2.4.* To simplify notations we set

$$X = X(t,x), \quad u = u(t, X(t,x)), \quad \varphi = \varphi(t, X(t,x)), \quad z_k = z_k(t), \quad \text{etc.}$$

For $t \in [0, T^*(x))$ we compute (assuming enough regularity on $\varphi$, see otherwise [11] for the regularization argument)

$$\begin{aligned} F_k' &= \sum_{j=1}^N \left\langle \frac{\gamma_j}{2\pi} \frac{X - z_j}{|X - z_j|^2}, \sum_{m \neq j} \gamma_m K(X - z_m) + u - \dot{z}_j \right\rangle \\ &\quad + \partial_t \varphi + \langle \dot{X}, \nabla \varphi \rangle + \langle \dot{X}, \dot{z}_k^\perp \rangle + \langle X, \ddot{z}_i^\perp \rangle \\ &= \sum_{j=1}^N \left\langle \frac{\gamma_j}{2\pi} \frac{X - z_j}{|X - z_j|^2}, \sum_{m \neq j} \gamma_m K(X - z_m) \right\rangle + \sum_{j=1}^N \left\langle \frac{\gamma_j}{2\pi} \frac{X - z_j}{|X - z_j|^2}, u(t,X) - \dot{z}_j \right\rangle \\ &\quad + \partial_t \varphi + \langle \dot{X}, \nabla \varphi_\varepsilon \rangle + \langle \dot{X}, \dot{h}_k^\perp \rangle + \langle X, \ddot{h}_k^\perp \rangle . \end{aligned}$$

We observe that

$$\sum_{j=1}^N \left\langle \frac{\gamma_j}{2\pi} \frac{X - z_j}{|X - z_j|^2}, \sum_{m \neq j} \frac{\gamma_m}{2\pi} \frac{(X - z_m)^\perp}{|X - z_m|^2} \right\rangle = \sum_{j=1}^N \left\langle \frac{\gamma_j}{2\pi} \frac{X - z_j}{|X - z_j|^2}, \sum_{m=1}^N \frac{\gamma_m}{2\pi} \frac{(X - z_m)^\perp}{|X - z_m|^2} \right\rangle = 0.$$

Thus we find

$$\begin{aligned} F_k' = \left\langle \frac{\gamma_k}{2\pi} \frac{X - z_k}{|X - z_k|^2}, u - \dot{z}_k \right\rangle + \sum_{j \neq k} \left\langle \frac{\gamma_j}{2\pi} \frac{X - z_j}{|X - z_j|^2}, u - \dot{z}_j \right\rangle \\ + \partial_t \varphi + \langle \dot{X}, \nabla \varphi \rangle + \langle \dot{X}, \dot{z}_k^\perp \rangle + \langle X, \ddot{z}_k^\perp \rangle. \end{aligned}$$

Next, since $X$ is a classical solution of the ODE with field $b$ defined in (2.7), we have

$$\frac{\gamma_k}{2\pi} \frac{X - z_k}{|X - z_k|^2} = -\dot{X}^\perp + u^\perp - \sum_{j \neq k} \frac{\gamma_j}{2\pi} \frac{X - z_j}{|X - z_j|^2}, \tag{2.12}$$

thus

$$\begin{aligned} F_k' &= -\langle \dot{X}^\perp, u - \dot{z}_k \rangle + \langle u^\perp, u - \dot{z}_k \rangle \\ &\quad - \sum_{j \neq k} \left\langle \frac{\gamma_j}{2\pi} \frac{X - z_j}{|X - z_j|^2}, u(t,X) - \dot{z}_k \right\rangle + \sum_{j \neq k} \left\langle \frac{\gamma_j}{2\pi} \frac{X - z_j}{|X - z_j|^2}, u - \dot{z}_j \right\rangle \\ &\quad + \partial_t \varphi + \langle \dot{X}, \nabla \varphi \rangle + \langle \dot{X}, \dot{z}_k^\perp \rangle + \langle X, \ddot{z}_k^\perp \rangle \\ &= \Big[ - \langle \dot{X}^\perp, u \rangle + \langle \dot{X}, \nabla \varphi \rangle \Big] + \Big[ \langle \dot{X}^\perp, \dot{z}_k \rangle + \langle \dot{X}, \dot{z}_k^\perp \rangle \Big] \\ &\quad - \langle u^\perp, \dot{z}_k \rangle + \sum_{j \neq k} \left\langle \frac{\gamma_j}{2\pi} \frac{X - z_j}{|X - z_j|^2}, \dot{z}_k - \dot{z}_j \right\rangle + \partial_t \varphi + \langle X, \ddot{h}_k^\perp \rangle. \end{aligned}$$

Finally, plugging the equality $\nabla^\perp \varphi = u$, we end up with

$$F_k' = -\langle u^\perp, \dot{z}_k \rangle + \sum_{j \neq k} \left\langle \frac{\gamma_j}{2\pi} \frac{X - z_j}{|X - z_j|^2}, \dot{z}_k - \dot{z}_j \right\rangle + \partial_t \varphi + \langle X, \ddot{z}_k^\perp \rangle.$$

Now, we recall (2.3), (2.4) and (2.6), which also imply that $|\ddot{z}_k| \leq C_T$. It follows that one can prove the following estimate on the flow:

$$\sup_{t \in [0,T]} |X(t,x)| \leq |x| + C_T. \tag{2.13}$$

Finally, it is also proved in [11], adapting the arguments of [13] for the fixed point vortex system, that

$$\|\partial_t \varphi\|_{L^\infty} \leq C_T.$$

Therefore we have proved the estimate for $F_k$. □

**Remark 2.5.** If we consider a given set of $C^2$ trajectories $\{z_k\}$ on $[0, T]$, the previous computations are still valid without using the dynamics of the point vortices. Then the constant in the estimate for $F_k$ depends on $\|z_k\|_{W^{2,\infty}}$.

**Remark 2.6.** Since $\omega(t,\cdot) = X(t,\cdot)_{\#}\omega_0$, (2.13) implies that $\omega(t,\cdot)$ is supported on some $B(0, R_0 + C_T)$ on $[0,T]$.

Thanks to Proposition 2.4 we prove the following lower bound:

**Corollary 2.7.** *There exists $K_T > 3$, depending only on $T$ and on the initial conditions, satisfying the following property. If $|X(t_0,x) - z_k(t_0)| < d_T/K_T$ for some $t_0 \in [0,T(x))$ and some $k = 1,\dots,N$, then $T(x) = T$. Moreover,*

$$\lambda_T|x - z_k(0)| \le |X(t,x) - z_k(t)| < \frac{d_T}{3}, \quad \forall\, t \in [0,T].$$

*Here $\lambda_T$ denotes a constant depending only on the initial conditions and on $T$. We recall that $d_T$ denotes the minimal distance between the point vortices defined in* (2.3).

This shows first that collision between the flow trajectory and the point vortices never occurs. This shows also that if the flow trajectory is at some time quite close to one of the point vortices, it never gets close to one other point vortex.

*Proof of Corollary 2.7.* First, Remark 2.6 implies that $|\varphi(t,x)| \le C_T \ln(1+|x|)$. Thus, by (2.13), this implies that

$$|\varphi(t,X(t,x))| \le C_T, \quad \forall\, t \in [0,T(x)), \quad \text{for a.e. } x \in \operatorname{supp}(\omega_0). \tag{2.14}$$

Next, we let $t_1 \in [0,T(x))$ be maximal such that

$$|X(t,x) - z_k(t)| < \frac{d_T}{3}, \quad \text{for } t \in [t_0,t_1).$$

In particular, we have

$$|X(t,x) - z_j(t)| > \frac{2d_T}{3} \quad \text{for } j \neq k.$$

Therefore, in view of Proposition 2.4, we have $|F_k'| \le C_T$ on $[t_0,t_1)$. Integrating this on $[t_0,t_1)$, using that

$$\sum_{j\neq k} |\ln|X(t,x) - z_j(t)|| \le C_T \quad \text{on } [t_0,t_1),$$

and using (2.14), we find:

$$|\ln|X(t_0,x) - z_k(t_0)| - \ln|X(t_1,x) - z_k(t_1)|| \le C_T. \tag{2.15}$$

We assume by contradiction that $t_1 < T(x)$. Therefore, $|X(t_1,x) - z_k(t_1)| = d_T/3$. Estimate (2.15) then yields

$$\ln\left(\frac{K_T}{3}\right) \le C_T,$$

a contradiction if $K_T$ is sufficiently large. So $t_1 = T(x)$. Integrating again the inequality $|F_k'| \le C_T$ on $[t_0,T(x))$, we obtain

$$\ln|X(t_0,x) - z_k(t_0)| \le \ln|X(t,x) - z_k(t)| + C_T \quad \text{on } [t_0,T(x)),$$

while we have just proved that

$$|X(t,x) - z_j(t)| > \frac{2d_T}{3} \quad \text{on } [t_0,T(x)).$$

This proves that no collision occurs on $[t_0,T(x))$, so $T(x) = T$.

By the same arguments, we finally show that (2.15) holds for $t_0$ and $t_1$ replaced by 0 and by any $t \in [0,T]$. Thus

$$\ln|x - z_k(0)| \le \ln|X(t,x) - z_k(t)| + C_T, \quad \forall\, t \in [0,T],$$

hence the conclusion follows. □

**Corollary 2.8.** *We have $T(x) = T$ for a.e. $x \in \operatorname{supp}(\omega_0)$. In particular, the proof of Theorem 1.1 is completed.*

*Proof of Corollary 2.8.* This is a direct consequence of Corollary 2.7. □

## 3. Proofs of Theorems 1.4 and 1.3

### 3.1. Proof of Theorem 1.4

The proof of Theorem 1.4 relies crucially on the lower bound of Corollary 2.7. The details are provided in [11].

### 3.2. Proof of Theorem 1.3. Case of one point vortex.

The next paragraph is devoted to the proof of uniqueness. We assume first that $N = 1$ in order to simplify the presentation.

Let $(\omega, z)$ and $(\widetilde{\omega}, \widetilde{z})$ be two solutions of (1.5) with initial datum satisfying the assumption of Theorem 1.3. So, Theorem 1.4 holds for both solutions: $\omega(t, \cdot\,)$ and $\widetilde{\omega}(t, \cdot\,)$ remain constant in a neighborhood of $z(t)$ and $\tilde{z}(t)$ for all $t \geq 0$.

We have $u - \widetilde{u} = K * (\omega - \widetilde{\omega})$ with $\int(\omega - \widetilde{\omega}) = \int \omega_0 - \int \omega_0 = 0$. Morever, $\omega(t, \cdot\,) - \widetilde{\omega}(t, \cdot\,)$ is complactly supported by Remark 2.6. Thus $u(t, \cdot\,) - \widetilde{u}(t, \cdot\,) \in L^2(\mathbb{R}^2)$, see e.g. [12, Proposition 3.3]. We may consider the quantity

$$D(t) = \|u(t, \cdot\,) - \widetilde{u}(t, \cdot\,)\|_{L^2}^2 + |z(t) - \widetilde{z}(t)|^2 + |\dot{z}(t) - \dot{\widetilde{z}}(t)|, \quad t \in [0, T].$$

Our purpose is to establish a Gronwall inequality for $D$, which will imply $D \equiv 0$ on $[0, T]$ since $D(0) = 0$, and therefore uniqueness.

We first use the estimates derived for (1.3) in [10, Subsection 3.4] for the quantity $\|u(t, \cdot\,) - \widetilde{u}(t, \cdot\,)\|_{L^2}^2$. These estimates rely:

- On the one hand, on the PDE satisfied by $u - \tilde{u}$, see [10, Subsection 3.4].
- On the other hand, on the *harmonicity* of $u(t, \cdot\,) - \tilde{u}(t, \cdot\,)$ in the neighborhood of $z(t)$ and $\tilde{z}(t)$, which is a direct consequence of Theorem 1.4. This enables to derive estimates for the $W^{1,\infty}$ norm of $u(t, \cdot\,) - \widetilde{u}(t, \cdot\,)$ in the neighborhood of $z(t)$ and $\tilde{z}(t)$ in terms of the $L^2$ norm thanks to the mean-value theorem:

  **Lemma 3.1.** *Let $y(t) = (z(t) + \tilde{z}(t))/2$. As long as $|z - \tilde{z}| < \delta$, $u(t, \cdot\,) - \widetilde{u}(t, \cdot\,)$ is harmonic on $B(y(t), \delta/2)$. Thus,*

  $$\|u(t, \cdot\,) - \widetilde{u}(t, \cdot\,)\|_{L^\infty(B(y(t),\delta/4))} \leq C\|u(t, \cdot\,) - \widetilde{u}(t, \cdot\,)\|_{L^2},$$
  $$\|\nabla u(t, \cdot\,) - \nabla\widetilde{u}(t, \cdot\,)\|_{L^\infty(B(y(t),\delta/4))} \leq C\|u(t, \cdot\,) - \widetilde{u}(t, \cdot\,)\|_{L^2},$$

  *for a constant $C$.*

Thanks to these ingredients, the estimate (3.9) in [10] yields for all $p \geq 2$ and as long as $D < \min(1, \delta^2)$,

$$\|u(t, \cdot\,) - \widetilde{u}(t, \cdot\,)\|_{L^2}^2 \leq C \int_0^t \Big(r(\tau) + \sqrt{r(\tau)}\varphi(\sqrt{r(\tau)}) + p\, r(\tau)^{1-1/p}\Big)\, \mathrm{d}\tau,$$

where

$$r(t) = \|u(t, \cdot\,) - \widetilde{u}(t, \cdot\,)\|_{L^2}^2 + |z(t) - \widetilde{z}(t)|^2$$

and

$$\varphi(\tau) = \tau|\ln \tau|.$$

Since $r(t) \leq D(t)$ and $\tau\varphi(\tau) \leq \varphi(\tau^2)$, $\tau \leq \varphi(\tau)$ for $\tau \leq 1$ and $\varphi(\tau) \leq p\tau^{1-1/p}$, we get for all $p \geq 2$, as long as $D < \min(1, \delta^2)$,

$$\|u(t, \cdot\,) - \widetilde{u}(t, \cdot\,)\|_{L^2}^2 \leq C\, p \int_0^t D(\tau)^{1-1/p}\, \mathrm{d}\tau. \tag{3.1}$$

We turn next to the estimate for the point vortices:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}|z - \widetilde{z}|^2 + \frac{\mathrm{d}}{\mathrm{d}t}|\dot{z} - \dot{\widetilde{z}}|^2 &= 2\langle z - \widetilde{z}, \dot{z} - \dot{\widetilde{z}}\rangle - 2\frac{\gamma}{m}\langle \dot{z} - \dot{\widetilde{z}}, u(t, z)^\perp - \widetilde{u}(t, \widetilde{z})^\perp\rangle \\ &\leq 2D + 2\frac{\gamma}{m}\sqrt{D}|u(t, z) - u(t, \widetilde{z})| + 2\frac{\gamma}{m}\sqrt{D}|(u - \widetilde{u})(t, \widetilde{z})|.\end{aligned}$$

Following the proof of [10, Proposition 3.10], we use again Lemma 3.1 and the LogLipschitz estimate (2.5) for $u$ to get that for all $p \geq 2$,

$$\frac{\mathrm{d}}{\mathrm{d}t}|z-\widetilde{z}|^2 + \frac{\mathrm{d}}{\mathrm{d}t}|\dot{z}-\dot{\widetilde{z}}|^2 \leq C\varphi(D) \leq CD^{1-1/p}, \quad \text{as long as } D < \min(1,\delta^2). \tag{3.2}$$

Finally, gathering (3.1) and (3.2), we find

$$D(t) \leq C\,p \int_0^t D(\tau)^{1-1/p}\,\mathrm{d}\tau, \quad \forall\, p \geq 2.$$

So by letting $p \to +\infty$, we conclude by usual arguments (see [12, Chapter 8] that $D \equiv 0$ on $[0,T]$.

### 3.3. Case of several point vortices

Finally, Theorem 1.3 follows easily by adapting the proof above to the case of several points. We refer to [16, Theorem 2.1, Chapter 2] for the details.

## 4. Some additional properties

We conclude this note by presenting a few well-known conservation properties for (1.3) and (1.5):

**Proposition 4.1.** *Let $(\omega, \{z_k\})$ be a weak solution to (1.3) on $[0,T]$. The following quantities are conserved:*

- *The energy,*

$$\mathcal{H}_0 = \frac{1}{2\pi}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \ln|x-y|\omega(t,y)\omega(t,x)\,\mathrm{d}x\,\mathrm{d}y + \frac{1}{\pi}\sum_{k=1}^{N}\gamma_k\int_{\mathbb{R}^2}\ln|x-z_k(t)|\omega(t,x)\,\mathrm{d}x + \sum_{j\neq k}\frac{\gamma_k\gamma_j}{2\pi}\ln|z_k(t)-z_j(t)|.$$

- *The momentum,*

$$\mathcal{I}_0 = \int_{\mathbb{R}^2}|x|^2\omega(t,x)\,\mathrm{d}x + \sum_{k=1}^{N}\gamma_k|z_k(t)|^2.$$

For the case with massive point vortices we have the analogous conservation properties:

**Proposition 4.2.** *Let $(\omega, \{z_k\})$ be a weak solution to (1.5) on $[0,T]$. The following quantities are conserved:*

- *The energy,*

$$\mathcal{H}_0 = \frac{1}{2\pi}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \ln|x-y|\omega(t,y)\omega(t,x)\,\mathrm{d}x\,\mathrm{d}y + \frac{1}{\pi}\sum_{k=1}^{N}\gamma_k\int_{\mathbb{R}^2}\ln|x-z_k(t)|\omega(t,x)\,\mathrm{d}x + \sum_{j\neq k}\frac{\gamma_k\gamma_j}{2\pi}\ln|z_k(t)-z_j(t)| - \sum_{k=1}^{N}m_k|\dot{z}_k(t)|^2.$$

- *The momentum,*

$$\mathcal{I}_0 = \int_{\mathbb{R}^2}|x|^2\omega(t,x)\,\mathrm{d}x + \sum_{k=1}^{N}\gamma_k|z_k(t)|^2 - 2\sum_{k=1}^{N}m_k z_k(t)^{\perp}\cdot\dot{z}_k(t).$$

In both cases, we infer that when the vorticity and circulations have the same sign, then uniform in time lower and upper bounds hold:

**Corollary 4.3.** *Assume that*

$$\omega_0 \geq 0, \ \textit{a.e. on } \mathbb{R}^2, \quad \gamma_k > 0, \ k = 1, \dots, N.$$

*Let* $(\omega, \{h_k\})$ *be any corresponding weak solution to* (1.3) *or* (1.5) *on* $[0,T]$*. Then there exists* $C > 0$ *and* $d > 0$*, depending only on the initial conditions, but not on* $T$*, such that*

$$\sup_{t\in[0,T]} \left(|\dot{z}_k(t)|^2 + |z_k(t)|^2\right) \leq C$$

*and*

$$\min_{t\in[0,T]} \min_{j\neq k} |z_j(t) - z_k(t)| \geq d.$$

*Proof of Corollary 4.3.* We refer to [11] for the proof. □

## References


[1] L. Ambrosio, "Transport equation and Cauchy problem for non-smooth vector fields", in *Calculus of variations and nonlinear partial differential equations*, Lecture Notes in Mathematics, vol. 1927, Springer, 2008, p. 1-42.

[2] ———, "Well posedness of ODE's and continuity equations with nonsmooth vector fields, and applications", preprint, 2017.

[3] G. Crippa & C. De Lellis, "Estimates and regularity results for the DiPerna–Lions flow", *J. Reine Angew. Math.* **616** (2008), p. 15-46.

[4] G. Crippa, L. M. C. Filho, E. Miot & H. J. Nussenzveig Lopes, "Flows of vector fields with point singularities and the vortex-wave system", *Discrete Contin. Dyn. Syst.* **36** (2016), no. 5, p. 2405-2417.

[5] R. J. DiPerna & P.-L. Lions, "Ordinary differential equations, transport theory and Sobolev spaces", *Invent. Math.* **98** (1989), no. 3, p. 511-547.

[6] O. Glass, C. Lacave & F. Sueur, "On the motion of a small body immersed in a two dimensional incompressible perfect fluid", *Bull. Soc. Math. Fr.* **142** (2014), no. 3, p. 489-536.

[7] ———, "On the motion of a small light body immersed in a two dimensional incompressible perfect fluid with vorticity", *Commun. Math. Phys.* **341** (2016), no. 3, p. 1015-1065.

[8] E. Hölder, "Über die unbeschränkte Fortsetzbarkeit einer stetigen ebenen Bewegung in einer unbegrenzten inkompressiblen Flüssigkeit", *Math. Z.* **37** (1993), p. 727-738.

[9] D. Iftimie, M. C. Lopes Filho & H. J. Nussenzveig Lopes, "Two dimensional incompressible ideal flow around a small obstacle", *Commun. Partial Differ. Equations* **28** (2003), no. 1-2, p. 349-379.

[10] C. Lacave & E. Miot, "Uniqueness for the vortex-wave system when the vorticity is initially constant near the point vortex", *SIAM J. Math. Anal.* **41** (2009), no. 3, p. 1138-1163.

[11] ———, "The vortex-wave system with gyroscopic effects", https://arxiv.org/abs/1903.01714, 2019.

[12] A. J. Majda & A. L. Bertozzi, *Vorticity and incompressible flow*, Cambridge Texts in Applied Mathematics, vol. 27, Cambridge University Press, 2002.

[13] C. Marchioro, "On the Euler equations with a singular external velocity field", *Rend. Semin. Mat. Univ. Padova* **84** (1990), p. 61-69.

[14] C. Marchioro & M. Pulvirenti, “On the vortex-wave system”, in *Mechanics, analysis, and geometry: 200 years after Lagrange*, North-Holland Delta Series, North-Holland, 1991, p. 79-95.

[15] ———, *Mathematical Theory of Incompressible Nonviscous Fluids*, Applied Mathematical Sciences, vol. 96, Springer, 1994.

[16] E. Miot, “Quelques problèmes relatifs à la dynamique des points vortex dans les équations d’Euler et de Ginzburg-Landau complexe”, PhD Thesis, Université Pierre et Marie Curie - Paris VI (France), 2009.

[17] A. Moussa & F. Sueur, “On a Vlasov–Euler system for 2D sprays with gyroscopic effects”, *Asymptotic Anal.* **81** (2013), no. 1, p. 53-91.

[18] V. N. Starovoǐtov, “Solvability of the problem of motion of concentrated vortices in an ideal fluid”, *Din. Splosh. Sredy* **85** (1988), p. 118-136.

[19] ———, “Uniqueness of a solution to the problem of evolution of a point vortex”, *Sib. Math. J.* **35** (1994), no. 3, p. 625-630.

[20] E. M. Stein, *Harmonic Analysis: Real-Variable Methods, Orthogonality, and Oscillatory Integrals*, Princeton Mathematical Series, vol. 43, Princeton University Press, 1993.

[21] W. Wolibner, “Un théorème sur l’existence du mouvement plan d’un fluide parfait homogène, incompressible, pendant un temps infiniment long”, *Math. Z.* **37** (1933), no. 1, p. 698-726.

[22] V. I. Yudovich, “Non-stationary flows of an ideal incompressible fluid”, *Zh. Vychisl. Mat. Mat. Fiz.* **3** (1963), no. 6, p. 1032-1066, English translation in *USSR Comput. Math. Math. Phys.* **3** (1963), no. 6, p. 1407-1456.

Evelyne Miot
CNRS-Université Grenoble Alpes
Institut Fourier UMR 5582
100, rue des mathématiques
38610 Gières
France
evelyne.miot@univ-grenoble-alpes.fr