Journées

ÉQUATIONS AUX DÉRIVÉES PARTIELLES

Port d'Albret, 7 juin–11 juin 2010

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J. É. D. P. (2010), Exposé n° VII, 13 p.

<http://jedp.cedram.org/item?id=JEDP_2010____A7_0>

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GDR 2434 (CNRS)

Electromagnetic Schrödinger flow: multiplier methods for dispersion

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Abstract

We show a list of results which have been recently obtained about dispersive properties of the electromagnetic Schrödinger flow. We introduce a general philosophy, based on multiplier technique, which permits to detect the bad parts of an electromagnetic potential which can possibly affect the dispersion.

1. Introduction

In the research about dispersive equations, a lot of mathematical tools have been developed to understand dispersive phaenomena. Dispersive equations involve some relevant models, which have been deeply investigated in the last years; among them, we mention the Schrödinger equation, the wave and Klein-Gordon equations, the Dirac equation. Motivated by solving some nonlinear models related to the previous examples, a strong effort was devoted to understand the dispersive properties of the respective linear evolution operators, which can be translated in terms of linear a priori estimates for solutions. In this paper, we take the Schrödinger equation as a model; however, we emphasize that the following discussion can be rephrased in terms of the other equations we mentioned before.

Consider the free Schrödinger equation

$$\begin{cases} \partial_t u(t,x) = i\Delta u(t,x) \\ u(0,x) = u_0(x), \end{cases} \tag{1.1}$$

where $u = u(t,x) : \mathbb{R}^{1+d} \to \mathbb{C}$. It is well known that, given any initial datum $u_0 \in L^2(\mathbb{R}^d)$, the unique solution $u \in \mathcal{C}(\mathbb{R}; L^2(\mathbb{R}^d))$ of (1.1) is given by

$$u(t,x) = \frac{1}{(4\pi i t)^{\frac{d}{2}}} \int e^{-\frac{ix\cdot y}{2t}} e^{\frac{i|y|^2}{4t}} u_0(y)\, dy =: K_t * u_0(x), \tag{1.2}$$

Keywords: electric potentials, magnetic potentials, virial identities, Schrödinger operators, spectral theory.

where $K_t(x) = (4\pi it)^{-\frac{d}{2}} e^{i\frac{|x|^2}{4t}}$. By the representation formula (1.2), it turns out immediately that the following *time decay estimates* hold:

$$\|u(t,\cdot)\|_{L^p} \lesssim |t|^{-d\left(\frac{1}{2}-\frac{1}{p}\right)} \|u_0\|_{L^{p'}}, \qquad p \geq 2 \qquad \frac{1}{p}+\frac{1}{p'}=1, \tag{1.3}$$

where u is given by (1.2), for any $t \neq 0$, where the symbol $\lesssim$ means that the inequality holds modulo a positive constant which does not depend on t and u_0.

The time decay estimates (1.3) imply some smoothing effects, in terms of L^p-integrability, for free Schrödinger solutions, which are well known as *Strichartz estimates*, stating that

$$\|e^{it\Delta} f\|_{L_t^p L_x^q} \lesssim \|f\|_{L_x^2}, \tag{1.4}$$

for any couple (p,q) satisfying the admissibility condition

$$\frac{2}{p} = \frac{d}{2} + \frac{d}{q}, \qquad \begin{cases} p \geq 4 & \text{if } d = 1 \\ p > 2 & \text{if } d = 2 \\ p \geq 2 & \text{if } d \geq 3. \end{cases}$$

For a standard proof of (1.4) in the non-endpoint case, see e.g. [14], while for the endpoint case the proof is given in [20].

There is moreover another family of dispersive estimates, which we usually refer to as *weak dispersive estimates*, involving for example the Morawetz estimate

$$\int\int \frac{|\partial_\tau e^{it\Delta} f|^2}{|x|}\, dx\, dt \lesssim \|f\|^2_{\dot{H}^{\frac{1}{2}}} \tag{1.5}$$

and the local smoothing estimate

$$\sup_{R>0} \frac{1}{R} \int \int_{|x|<R} |\nabla e^{it\Delta} f|^2\, dx\, dt \lesssim \|f\|^2_{\dot{H}^{\frac{1}{2}}}, \tag{1.6}$$

where ∂_τ is the tangential component of the gradient to the sphere and $\|\cdot\|_{\dot{H}^{1/2}} = \||D|^{1/2}\cdot\|_{L^2}$. An estimate of the same type of (1.5) was proved by C. Morawetz in [22] first, for the Klein-Gordon equation (without gain of derivatives, which is not true in that case); the kind of techniques we we will use in the sequel refers, in fact, to the original Morawetz' paper. The local smoothing (1.6) was proved independently in [3, 27, 30].

Starting from the pioneer paper [22], multiplier methods have been developed to prove a priori estimates of the type (1.5), (1.6). Due to this, weak dispersive estimates are the most adaptable to be proved for some perturbations of the Schrödinger equation (e.g. variable coefficients, NLS). The aim of this paper is to introduce the above mentioned multiplier techniques, with particular interest to the model of Schrödinger equation with an external electromagnetic field. We will show how estimates (1.5), (1.6) can be proved; finally, we will explain how Strichartz estimates for the electromagnetic Schrödinger equation can be proved once weak dispersion is known.

2. Viral-type identities for the free Schrödinger equation

Let us consider the free Schrödinger equation (1.1), where the solution u is given by (1.2). Let $T : L^2 \to L^2$ be a linear operator and consider the quantity

$$\Theta(t) = (u, Tu) = \int u(t)\overline{Tu(t)}\, dx.$$

A formal computation gives

$$\frac{d}{dt}\Theta(t) = i(u, [\Delta, T]u) \tag{2.1}$$

$$\frac{d^2}{dt^2}\Theta(t) = -(u, [\Delta, [\Delta, T]]u), \tag{2.2}$$

where the brackets $[\cdot, \cdot]$ denote the commutator. Identity (2.1) is in fact the Eisenberg equation. When T is the multiplication operator for a function $\phi : \mathbb{R}^n \to \mathbb{R}$, the commutators can be written explicitly as differential operators with variable coefficients depending on ϕ and its derivatives; integration by parts leads to the following *virial-type* identities:

Lemma 2.1. *Let $\phi : \mathbb{R}^n \to \mathbb{R}$ be a positive function and define $\Theta(t) = \int \phi|u|^2\, dx$, where u is given by* (1.2)*. The following formal identities hold:*

$$\frac{d}{dt}\Theta(t) = 2\Im \int \overline{u}\nabla\phi \cdot \nabla u\, dx \tag{2.3}$$

$$\frac{d^2}{dt^2}\Theta(t) = 4\int \nabla u D^2\phi \nabla\overline{u}\, dx - \int |u|^2\Delta^2\phi\, dx, \tag{2.4}$$

where $D^2\phi$ is the Hessian matrix of ϕ and $\Delta^2\phi = \Delta(\Delta\phi)$ is the bi-laplace.

We now pass to some relevant examples of multipliers ϕ.

2.1. Virial Multiplier

Let us consider the **virial multiplier** $\phi = |x|^2$. In this case, (2.4) reads as follows:

$$\frac{d^2}{dt^2}\int |x|^2|u|^2\, dx = 8\int |\nabla u|^2\, dx = 16E, \tag{2.5}$$

being E the energy, which is an invariant of the motion. Equation (2.5) is the well known *virial identity.* It affirms, in particular, that the position $\|xu\|_{L^2}$ is a strictly convex function of time, if u is nontrivial. Since it is also positive, it tends to infinity as $t \to \pm\infty$, which means that free Schrödinger particles have the propagation property. This convexity property is a kind of characterizing fact for dispersive equations.

2.2. Morawetz Multiplier

Let us now consider the **Morawetz multiplier** $\phi = |x|$, in dimension $d \geq 3$. Due to the following identity

$$\nabla u D^2\phi \nabla\overline{u} = \phi''|\partial_r u|^2 + \frac{\phi'}{|x|}|\partial_\tau u|^2,$$

which holds for radial functions ϕ, by (2.3) and (2.4) we obtain

$$2\Im\frac{d}{dt}\int \overline{u}\partial_r u\,dx = 4\int \frac{|\partial_\tau u|^2}{|x|}\,dx - \int |u|^2\Delta^2|x|\,dx.$$

Since $\Delta^2|x| \leq 0$ if $d \geq 3$, and since by interpolation we have $|\int \overline{u}\partial_r u| \leq \|u\|_{\dot{H}^{1/2}}$, integrating in time the last identity and using the conservation of the $\dot{H}^{\frac{1}{2}}$-norm we get the Morawetz estimate

$$\int\int \frac{|\partial_\tau u|^2}{|x|}\,dx\,dt \lesssim \|u(0)\|^2_{\dot{H}^{\frac{1}{2}}}. \tag{2.6}$$

Once again, the convexity of $\||x|^{1/2}u\|_{L^2}$ as a function of time was a crucial fact here.

2.3. Local smoothing multiplier

We now introduce a new multiplier, which mixes the previous two ones. Denote by $r = |x|$; for any $R > 0$, we need a function ϕ_R with the following behavior:

$$\phi_R(r) \simeq \begin{cases} r^2, & r \leq R \\ r, & r > R. \end{cases}$$

An explicit example is given by the following one: in dimension $d \geq 3$, for any $R > 0$, let us consider $\phi_R(r) = \int_0^r \phi_R'(s)\,ds$, with

$$\phi_R'(r) = \begin{cases} \frac{(d-1)r}{2nR}, & r \leq R \\ \frac{1}{2} - \frac{R}{2dr^{d-1}}, & r > R. \end{cases} \tag{2.7}$$

An explicit computation shows that

$$\phi_R''(r) = \begin{cases} \frac{1}{R}\cdot\frac{d-1}{2d}, & r \leq R \\ \frac{1}{R}\cdot\frac{R^d(d-1)}{2dr^d}, & r > R, \end{cases} \tag{2.8}$$

$$\Delta^2\phi_R = \begin{cases} -4\pi\delta_{x=0} - \frac{1}{R^2}\delta_{|x|=R}, & d = 3 \\ -\frac{d-1}{2}\delta_{|x|=R} - \frac{(d-1)(d-3)}{2r^3}\chi_{[1,+\infty)}, & d \geq 4. \end{cases} \tag{2.9}$$

Notice that $\Delta^2\phi < 0$ and $\phi', \phi'' \geq 0$, since $d \geq 3$. Plugging it into the virial identities (2.3), (2.4) and arguing as above, it gives the following estimates:

$$\sup_{R>0}\frac{1}{R}\int\int_{|x|<R}|\nabla u|^2\,dx\,dt + \sup_{R>0}\frac{1}{R^2}\int\int_{|x|=R}|u|^2\,d\sigma(x)\,dt \lesssim \|u(0)\|^2_{\dot{H}^{\frac{1}{2}}}. \tag{2.10}$$

This kind of multiplier was used in [23] first, for the Helmholtz equation, and then in [1] for the electric Schrödinger equation. The refined explicit version given by (2.7) was introduced in [12].

3. Electromagnetic potentials

A Schrödinger Hamiltonian with electromagnetic potential is an operator of the form

$$H = -(\nabla - iA)^2 + V,$$

where $A = (A^1, \dots, A^d) : \mathbb{R}^d \to \mathbb{R}^d$ and $V : \mathbb{R}^d \to \mathbb{R}$ are the magnetic and electric potential, respectively. The Cauchy problem for the electromagnetic Schrödinger equation is hence given by

$$\partial_t u = iHuu(0) = u_0. \tag{3.1}$$

It describes the interaction of a free particle with an external electromagnetic field. The electric field is given by $E = \nabla V$. Once identifying A with the linear 1-form $\omega = \sum_{j=1}^{d} A^j(x)dx_j$, the magnetic field is given by the differential $B = d\omega$. In the euclidean setting, we can also write explicitly $B = DA - DA^t$ as the antisymmetric component of the matrix gradient of A. In particular, since in dimension $d = 3$ 1-forms and 2-forms are identified, we also have that

$$Bv = (DA - DA^t)v = \operatorname{curl} A \times v, \qquad (d = 3)$$

for all $v \in \mathbb{R}^3$, where the cross denotes the vectorial product in $\mathbb{R}^3$.

From now on, we will always need the following regularity assumptions on the Hamiltonian H:

(H1) The Hamiltonian $H_A = -\nabla_A^2$ is essentially self-adjoint on $L^2(\mathbb{R}^n)$, with form domain

$$D(H_A) = \left\{ f : f \in L^2, \int |\nabla_A f|^2 < \infty \right\}.$$

(H2) The potential V is a perturbation of H_A in the Kato-Rellich sense, i.e. there exists a small $\epsilon > 0$ such that

$$\|Vf\|_{L^2} \leq (1 - \epsilon)\|H_A f\|_{L^2} + C\|f\|_{L^2}, \tag{3.2}$$

for all $f \in D(H_A)$.

Assumptions (H1), (H2) have several consequences about the existence theory for equation (3.1). First of all, they imply the self-adjointness of H, by standard perturbation techniques; hence by the spectral theorem we can define the Schrödinger propagator $S(t) = e^{itH}$. Moreover we can define for any s the distorted norms

$$\|f\|_{\dot{\mathcal{H}}^s} = \|H^{\frac{s}{2}} f\|_{L^2}. \qquad \|f\|_{\mathcal{H}^s} = \|f\|_{L^2} + \|H^{\frac{s}{2}} f\|_{L^2}.$$

Since H and H^s commute with each other, for any $s \geq 0$, the Schrödinger propagator $S(t)$ satisfy the family of conservation laws

$$\|e^{itH} f\|_{\dot{\mathcal{H}}^s} = \|f\|_{\dot{\mathcal{H}}^s}, \qquad s \geq 0,$$

for all $t \in \mathbb{R}$. For the validity of (H1) and (H2) see e.g. [4], and the standard reference [21].

A relevant property of equation (3.1) is the gauge invariance. In fact, if u solves (3.1), then $v = e^{i\varphi}u$ solves the same equations, with the new potential $\widetilde{A} = A + \nabla\varphi$. Notice that the magnetic field B does not change if we sum a gradient to the potential. Hence we are interested in giving results which respect this property, namely we need to state theorems in which the assumptions do not depend on the gauge choice. A relevant electromagnetic potential is the so-called Coulomb-type one. They are homogeneous potentials of degree -1 and -2 (respectivly the magnetic and the electric one); a model is

$$A = |x|^{-2}x, \qquad V = |x|^{-1}.$$

Notice that they leave equation (3.1) invariant under the natural scaling $u_\lambda(t, x) = u(t/\lambda^2, x/\lambda)$.

As we see in the following, the multiplier techniques we introduced before for the free Schrödinger equation, lead us to detect the bad parts of the potentials which possibly affect the dispersive properties of the flow e^{itH}. As for the electric part, the radial derivative $\partial_r V$ will play a crucial role; for the magnetic term, we need to introduce the quantity B_τ, defined as

$$B_\tau(x) = \frac{x}{|x|}B \qquad d \geq 2. \tag{3.3}$$

Notice that, in dimension $d = 3$, we can write

$$B_\tau = \frac{x}{|x|} \times \operatorname{curl} A. \tag{3.4}$$

Hence B_τ is a tangential projection of the vector field B on the sphere. We will usually refer to it as to the *trapping component* of the field B.

It is relevant to show examples of potentials A for which $B_\tau \equiv 0$; as it will be clear in the sequel, it is natural to call them *non-trapping magnetic potentials.* we will focus our attention on the 3D case.

Example 3.1. First we consider some singular potentials. Take

$$A = \frac{1}{x^2+y^2+z^2}(-y, x, 0) = \frac{1}{x^2+y^2+z^2}(x, y, z) \wedge (0, 0, 1). \tag{3.5}$$

We can check that

$$\nabla \cdot A = 0, \qquad B = -2\frac{z}{(x^2+y^2+z^2)^2}(x, y, z), \qquad B_\tau = 0.$$

Another (more singular) example is the following:

$$A = \left(\frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2}, 0\right) = \frac{1}{x^2+y^2}(x, y, z) \wedge (0, 0, 1). \tag{3.6}$$

Here we have $B = (0, 0, \delta)$, with δ denoting Dirac's delta function. Again we have $B_\tau = 0$.

Example 3.2. Now we show a natural generalization of the previous examples. Assume that $B = \operatorname{curl} A : \mathbb{R}^3 \to \mathbb{R}^3$ is known, and assume that $\operatorname{div} A = 0$; then we can reconstruct the potential A using the *Biot-Savart* formula

$$A(x) = \frac{1}{4\pi}\int \frac{x-y}{|x-y|^3} \wedge B(y)\, dy. \tag{3.7}$$

Assume now that $B_\tau = 0$, namely $x \wedge B(x) = 0$; by (3.7) we have

$$A(x) = \frac{x}{4\pi} \wedge \int \frac{B(y)}{|x-y|^3}\, dy. \tag{3.8}$$

To have $B_\tau = 0$ it is necessary $B(y) = g(y)\frac{y}{|y|}$, for some scalar function $g : \mathbb{R}^3 \to \mathbb{R}$. Since we want $A \neq 0$, g has not to be radial. As an example we consider

$$g(y) = h\left(\frac{y}{|y|} \cdot \omega\right)|y|^{-\alpha},$$

for some fixed $\omega \in S^2$, where h is homogeneous of degree 0 and $\alpha \in \mathbb{R}$; consequently, the vector field B is homogeneous of degree $-\alpha$. By (3.8) we have

$$A(x) = \frac{x}{4\pi} \wedge \int \frac{h\left(\frac{y}{|y|} \cdot \omega\right)}{|x-y|^3|y|^\alpha} y\, dy. \tag{3.9}$$

The potential A is homogenous of degree $1-\alpha$, and by symmetry we have that $A(\omega)=0$. These examples can be easily extended to higher dimensions.

The definition of B_τ and the previous examples appear e.g. in [12].

4. Weak dispersion for the electromagnetic flow

We can now state our main results about the dispersive properties of the electromagnetic Schrödinger flow e^{itH}. First of all, in analogy with Lemma 2.1 we have the following:

Theorem 4.1 (Virial for magnetic Schrödinger [12])**.** *Let $\phi:\mathbb{R}^n\to\mathbb{R}$ be a radial, real-valued multiplier, $\phi=\phi(|x|)$, and let*

$$\Theta(t)=\int_{\mathbb{R}^n}\phi|u|^2\,dx. \tag{4.1}$$

Let $u=e^{itH}u_0$ be a solution of the magnetic Schrödinger equation (3.1) *with initial datum $u_0\in L^2$, $H_Au_0\in L^2$; then, for any $t\in\mathbb{R}$, the following virial-type identity holds:*

$$\dot\Theta(t)=2\Im\int\phi'\overline{u}\nabla_Au\cdot\frac{x}{|x|}\,dx \tag{4.2}$$

$$\begin{aligned}\ddot\Theta(t)=&4\int_{\mathbb{R}^n}\nabla_AuD^2\phi\overline{\nabla_Au}\,dx-\int_{\mathbb{R}^n}|u|^2\Delta^2\phi\,dx\\&-2\int_{\mathbb{R}^n}\phi'\left(\partial_rV\right)|u|^2\,dx+4\Im\int_{\mathbb{R}^n}u\phi'B_\tau\cdot\overline{\nabla_Au}\,dx,\end{aligned} \tag{4.3}$$

where

$$\left(D^2\phi\right)_{jk}=\frac{\partial^2}{\partial x^j\partial x^k}\phi,\qquad\Delta^2\phi=\Delta(\Delta\phi),$$

for $j,k=1,\dots,n$, are respectively the Hessian matrix and the bi-Laplacian of ϕ, and $\nabla_A:=\nabla-iA$.

Notice that, in the free case $A\equiv V\equiv0$, the previous result coincides with the one in Lemma 2.1. The proof of Theorem 4.1 can be performed by a direct computation, for regular solutions $u\in\mathcal{H}^2$, and then completed for $\mathcal{H}^1$-solutions via a density argument; see [12] for the details.

Observe that only standard derivatives act on the multiplier ϕ in identity (4.3), while the covariant derivatives ∇_A only act on the solution u. This is due to the following Leibnitz rule:

$$\nabla_A(fg)=g\nabla_Af+f\nabla g.$$

This way to write the identity permits to use the same multipliers as in the free case, in order to get convexity for Θ; indeed, the terms involving ∂_rV and B_τ in (4.3) can be treated as perturbations of the free identity (2.4). We now specialize the previous identities to the above introduced multipliers.

Virial Multiplier. Using the virial multiplier $\phi=|x|^2$, we immediately obtain the following Corollary of Theorem 4.1, which is the analogous of the virial identity (2.5).

Corollary 4.2. *Let $u=e^{itH}u_0$ be a solution of the magnetic Schrödinger equation* (3.1) *with $f\in L^2$, $H_Af\in L^2$. Then the quantity*

$$Q(t)=\int_{\mathbb{R}^n}|x|^2|u|^2\,dx$$

satisfies the identity

$$\ddot{Q}(t) = 8\int_{\mathbb{R}^n} |\nabla_A u|^2 \, dx - 4\int_{\mathbb{R}^n} |x| \left(\partial_r V\right) |u|^2 \, dx + 8\Im \int_{\mathbb{R}^n} |x| u B_\tau \cdot \overline{\nabla_A u} \, dx. \qquad (4.4)$$

Consequently, the strict convexity of Q for nontrivial solutions of (3.1) is a matter of fact, if $B_\tau \equiv 0$ and $\partial_r V \leq 0$. When these conditions are satisfied, we say that V is **repulsive** and A is **non-trapping**.

Morawetz and local smoothing multipliers. We can now consider the sum of the Morawetz multiplier $\phi = |x|$ with the local smoothing multiplier ϕ_R given by (2.7), (2.8), (2.9). Plugging them into identities (4.2), (4.3), we can prove the two following Theorems, regarding weak dispersive properties of the group e^{itH}, respectively in dimension $d = 3$ and $d \geq 4$.

Theorem 4.3 (Weak dispersion for 3D Schrödinger [12])**.** *Let $d = 3$, and assume (H1), (H2); denote by*

$$\int_0^{+\infty} \rho^3 \sup_{|x|=\rho} |B_\tau|^2 \, d\rho = \left\| |x|^{\frac{3}{2}} B_\tau \right\|^2_{L^2_r L^\infty(S_r)} =: C_1$$

$$\int \rho^2 \sup_{|x|=\rho} \left| (\partial_r V)_+ \right| \, d\rho = \left\| |x|^2 \left(\partial_r V\right)_+ \right\|_{L^1_r L^\infty(S_r)} =: C_2,$$

and assume that

$$\frac{\sqrt{C_1} + \sqrt{C_1 + 2C_2}}{\sqrt{C_1 + 2C_2}} \left(\frac{\sqrt{C_1} + \sqrt{C_1 + 2C_2}}{2} \sqrt{C_1} + C_2 \right) \leq \frac{1}{2}. \qquad (4.5)$$

Then, for any solution u of (3.1) *with $f \in L^2$, $H_A f \in L^2$, the following estimate holds:*

$$\sup_{R>0} \frac{1}{R} \int_0^{+\infty} \int_{|x|\leq R} |\nabla_A u|^2 \, dx \, dt \leq C \|f\|^2_{\dot{\mathcal{H}}^{\frac{1}{2}}} \qquad (4.6)$$

for some $C > 0$. Moreover, if the strict inequality holds in (4.5)*, we also have*

$$\sup_{R>0} \frac{1}{R} \int_0^{+\infty} \int_{|x|\leq R} |\nabla_A u|^2 \, dx \, dt + \epsilon \int_0^{+\infty} \int_{\mathbb{R}^n} \frac{|\nabla_A^\tau u|^2}{|x|} \, dx \, dt \qquad (4.7)$$
$$+ \epsilon \sup_{R>0} \frac{1}{R^2} \int_0^{+\infty} \int_{|x|=R} |u|^2 \, d\sigma \, dt \leq C \|f\|^2_{\dot{\mathcal{H}}^{\frac{1}{2}}},$$

for some $\epsilon > 0$.

In higher dimension we prove the following Theorem.

Theorem 4.4 (Weak dispersion for higher dimensional Schrödinger [12])**.** *Let $d \geq 4$, and assume (H1), (H2); assume that*

$$|B_\tau(x)| \leq \frac{C_1}{|x|^2}, \qquad |V_r^+(x)| \leq \frac{C_2}{|x|^3}, \qquad C_1^2 + 2C_2 \leq \frac{2}{3}(d-1)(d-3), \qquad (4.8)$$

for all $x \in \mathbb{R}^n$. Then, for any solution of (3.1) *with $f \in L^2$, $H_A f \in L^2$, the following estimate holds:*

$$\sup_{R>0} \frac{1}{R} \int_0^{+\infty} \int_{|x|\leq R} |\nabla_A u|^2 \, dx \, dt \leq C \|f\|^2_{\dot{\mathcal{H}}^{\frac{1}{2}}},$$

for some $C > 0$. Moreover, if the strict inequality holds in (4.8)*, we also have*

$$\sup_{R>0} \frac{1}{R} \int_0^{+\infty} \int_{|x|\leq R} |\nabla_A u|^2 \, dx \, dt + \epsilon \int_0^{+\infty} \int_{\mathbb{R}^n} \frac{|\nabla_A^\tau u|^2}{|x|} \, dx \, dt \qquad (4.9)$$
$$+ \epsilon \frac{(d-1)(d-3)}{2} \int_0^{+\infty} \int \frac{|u|^2}{|x|^3} \, dx \, dt \leq C \|f\|^2_{\dot{\mathcal{H}}^{\frac{1}{2}}},$$

for some $\epsilon > 0$.

Remark 4.1. Conditions (4.5) and (4.8) have to be interpreted as the necessary (and probably sharp) smallness assumption on the bad parts of B and V, in order to obtain the same weak dispersive properties as in the free case. As for (4.5), notice that, when $C_1 = 0$ (non-trapping case), it reduces to $C_2 \leq \frac{1}{2}$, while if $C_2 = 0$ (repulsive case) it just requires $C_1 \leq \frac{1}{4}$.

Remark 4.2. Notice that the assumptions of the previous Theorems are given in terms of the fields; consequently, the previous results are invariant with respect to gauge transformations.

Remark 4.3. The analogous of the previous results, proved by the same techniques, for the purely electric Schrödinger flow, was proved in [1]. We also mention [8] and the references therein for a proof of local the local smoothing, also local in time, for the Schrödinger equation with unbounded electromagnetic potentials, in which the same kind of techniques are involved.

5. Endpoint Strichartz estimates

Strichartz estimates for the flow e^{itH} can be proved, by perturbative techniques, as a consequence of the weak dispersive Theorems of the previous section.

In the last years, a big effort was spent to prove Strichartz estimates for the Schrödinger equation with electromagnetic potential. In the purely electric case $A \equiv 0$ the literature is extensive and almost complete; we may cite among many others the papers [2], [16], [25]. It is now clear that in this case the decay $V(x) \sim 1/|x|^2$ is critical for the validity of Strichartz estimates; suitable counterexamples were constructed in [17]. In the magnetic case $A \neq 0$, the Coulomb decay $|A| \sim 1/|x|$ is also critical, as it was shown in the counterexamples produced in [11]. An intense research is ongoing concerning Strichartz estimates for the magnetic Schrödinger equation, see e.g. [7], [5], [6], [13]; see also [24] for a more general class of first order perturbations.

Actually, Theorems 4.3 and 4.4 suggest that a natural initial space in which to set the problem is $\dot{H}^{\frac{1}{2}}$. Let us fix the Coulomb gauge $\operatorname{div} A = 0$, which is in fact a not restrictive assumption. By the explicit expansion

$$H = -\Delta + 2iA \cdot \nabla_A - |A|^2 + V,$$

we see that the solution $u = e^{itH} u_0$ of (3.1) can be written in terms of the Duhamel formula as follows

$$e^{itH}\varphi = e^{it\Delta}\varphi + \int_0^t e^{i(t-s)\Delta} R(x, D) e^{itH}\varphi, ds, \qquad (5.1)$$

where the perturbative operator $R(x, D)$ is given by

$$R(x, D) = 2iA \cdot \nabla_A - |A|^2 + V. \qquad (5.2)$$

Now we recall the following estimate:

$$\left\| |D|^{\frac{1}{2}} \int_0^t e^{i(t-s)\Delta} F(s,\cdot) ds \right\|_{L^pL^q} \lesssim \sum_{j\in\mathbb{Z}} 2^{\frac{j}{2}} \|F_j\|_{L^2L^2}, \tag{5.3}$$

for any admissible couple (p,q) as above, where

$$F = \sum_{j\in\mathbb{Z}} F_j, \qquad \text{supp} F_j \subset \{2^j \leq |x| \leq 2^{j+1}\} \times \mathbb{R}$$

Estimate (5.3) was proved in [26] first; actually it follows by mixing the free Strichartz estimates for $T(t)$ with the dual of the local smoothing estimates which were proved independently by [3], [27] and [30]. In the paper [26] the endpoint estimate for $p=2$ is not proved (and indeed it predates the Keel-Tao paper [20]). The endpoint case $p=2$ in dimension $n \geq 3$ is a consequence of Lemma 3 in [18]. By (5.3) and free Strichartz, we can hence estimate

$$\left\| |D|^{\frac{1}{2}} e^{-itH} \varphi \right\|_{L^pL^q} \leq C \| |D|^{\frac{1}{2}} \varphi \|_{L^2} + \sum_{j\in\mathbb{Z}} 2^{\frac{j}{2}} \left\| \chi_j R(x,D) e^{itH} \varphi \right\|_{L^2L^2}, \tag{5.4}$$

where χ_j is the characteristic function of the ring $2^j \leq |x| \leq 2^{j+1}$.

Let us recall the definition of Kato norm.

Definition 5.1. Let $n \geq 3$. A measurable function $V(x)$ is said to be in the *Kato class* K_d provided

$$\lim_{r\downarrow 0} \sup_{x\in\mathbb{R}^d} \int_{|x-y|\leq r} \frac{|V(y)|}{|x-y|^{d-2}} dy = 0.$$

We shall usually omit the reference to the space dimension d and write simply K instead of K_d. The *Kato norm* is defined as

$$\|V\|_K = \sup_{x\in\mathbb{R}^d} \int \frac{|V(y)|}{|x-y|^{d-2}} dy.$$

The above computations suggest the proof of the following Theorem.

Theorem 5.2 ([9]). *Let $d \geq 3$. Given $A, V \in C^1_{loc}(\mathbb{R}^d \setminus \{0\})$, assume the operators $\Delta_A = -(\nabla - iA)^2$ and $H = -\Delta_A + V$ are selfadjoint and positive on L^2. Moreover assume that*

$$\|V_-\|_K < \frac{\pi^{\frac{d}{2}}}{\Gamma\left(\frac{d}{2}-1\right)} \tag{5.5}$$

for a sufficiently small $\epsilon > 0$ depending on A and

$$\sum_{j\in\mathbb{Z}} 2^j \sup_{|x|\sim 2^j} |A| + \sum_{j\in\mathbb{Z}} 2^{2j} \sup_{|x|\sim 2^j} |V| < \infty, \tag{5.6}$$

and the Coulomb gauge condition

$$\text{div } A = 0. \tag{5.7}$$

Finally, when $d = 3$, we assume that for some $M > 0$

$$\frac{\left(M+\frac{1}{2}\right)^2}{M} \| |x|^{\frac{3}{2}} B_\tau \|^2_{L^2_r L^\infty(S_r)} + (2M+1) \| |x|^2 (\partial_r V)_+ \|_{L^1_r L^\infty(S_r)} < \frac{1}{2}, \tag{5.8}$$

while for $n \geq 4$ we assume that

$$\| |x|^2 B_\tau(x) \|^2_{L^\infty} + 2 \| |x|^3 (\partial_r V)_+(x) \|_{L^\infty} < \frac{2}{3}(d-1)(d-3). \tag{5.9}$$

Then, for any Schrödinger admissible couple (p,q), the following Strichartz estimates hold:

$$\|e^{itH}\varphi\|_{L^pL^q} \leq C\|\varphi\|_{L^2}, \qquad \frac{2}{p}=\frac{d}{2}-\frac{d}{q}, \quad p\geq 2, \quad p\neq 2 \textit{ if } d=3. \tag{5.10}$$

In dimension $d=3$, we have the endpoint estimate

$$\||D|^{\frac{1}{2}}e^{itH}\varphi\|_{L^2L^6} \lesssim \|H^{\frac{1}{4}}\varphi\|_{L^2}. \tag{5.11}$$

The previous result was proved in [9], to which we remand for more details about the proof.

Remark 5.1. Let us remark that the regularity assumption $A, V \in C^1_{\text{loc}}(\mathbb{R}^d \setminus \{0\})$ is actually stronger than what we really require. For the validity of the Theorem, we just need to give meaning to inequalities (5.8), (5.9).

Remark 5.2. We emphasize that in Theorem 5.2 we do not require absence of resonances at energy zero, in contrast with [5], [6]. Indeed, this is possible thanks to the non-trapping and repulsivity conditions (5.8), (5.9); notice however that these conditions can be checked easily in concrete examples, which is not the case for the abstract assumption on resonances.

Remark 5.3. In order to perform the proof, we also need the following result about equivalence of standard and distorted Sobolev spaces, which we think is of outstanding interest:

Theorem 5.3 ([9])**.** *Let $d\geq 3$. Given $A\in L^2_{loc}(\mathbb{R}^n;\mathbb{R}^n)$, $V:\mathbb{R}^n\to\mathbb{R}$, assume the operators $\Delta_A=-(\nabla-iA)^2$ and $H=-\Delta_A+V$ are selfadjoint and positive on L^2. Moreover, assume that V_+ is of Kato class, V_- satisfies*

$$\|V_-\|_K < \frac{\pi^{\frac{d}{2}}}{\Gamma\left(\frac{d}{2}-1\right)}, \tag{5.12}$$

and

$$|A|^2+V\in L^{d/2,\infty}, \qquad A\in L^{d,\infty}. \tag{5.13}$$

Then the following estimate holds:

$$\|H^{1/4}f\|_{L^q} \leq C_q\||D|^{\frac{1}{2}}f\|_{L^q}, \qquad 1<q<2d, \qquad d\geq 3. \tag{5.14}$$

In addition we have the reverse estimate

$$\|H^{1/4}f\|_{L^q} \geq c_q\||D|^{\frac{1}{2}}f\|_{L^q}, \qquad \frac{4}{3}<q<4, \qquad d\geq 3. \tag{5.15}$$

Theorem 5.3 was proved in [9], by means of the Stein-Weiss interpolation Theorem and Barry Simon's diamagnetic inequalities. We remand to [9] for the detailed proof.

References

[1] Barceló, J.A., Ruiz, A., and Vega, L., Some dispersive estimates for Schrödinger equations with repulsive potentials *J. Funct. Anal.* **236** (2006), 1–24.

[2] Burq, N., Planchon, F., Stalker, J., and Tahvildar-Zadeh, S. Strichartz estimates for the wave and Schrödinger equations with potentials of critical decay, *Indiana Univ. Math. J.* **53**(6) (2004), 1665–1680.

[3] Constantin, P., and Saut, J.-C., Local smoothing properties of dispersive equations, *Journ. AMS* (1988), 413–439.

[4] Cycon, H.L., Froese, R., Kirsch, W., and Simon, B., *S*chrödinger Operators with Applications to Quantum Mechanics and Global Geometry Texts and Monographs in Physics, Springer Verlag Berlin Heidelberg New York (1987).

[5] Erdoğan, M. B., Goldberg, M., and Schlag, W., Strichartz and Smoothing Estimates for Schrödinger Operators with Almost Critical Magnetic Potentials in Three and Higher Dimensions, to appear on *Forum Math.*

[6] Erdoğan, M. B., Goldberg, M., and Schlag, W., Strichartz and smoothing estimates for Schrodinger operators with large magnetic potentials in $\mathbb{R}^3$, to appear on *J. European Math. Soc.*

[7] D'Ancona, P., and Fanelli, L., Strichartz and smoothing estimates for dispersive equations with magnetic potentials, *Comm. Part. Diff. Eqns.* **33** (2008), 1082–1112.

[8] P. D'Ancona, and L. Fanelli: Smoothing estimates for the Schrödinger equation with unbounded potentials, *Journ. Diff. Eq.* **246** (2009), 4552–4567.

[9] P. D'Ancona, L. Fanelli, L. Vega, and N. Visciglia: Endpoint Strichartz estimates for the magnetic Schrödinger equation, *J. Funct. Anal.* **258** (2010), 3227–3240.

[10] Fanelli, L., Non-trapping magnetic fields and Morrey-Campanato estimates for Schrödinger operators, textitJ. Math. Anal. Appl. **357** (2009), 1–14.

[11] L. Fanelli, and A. García: Counterexamples to Strichartz estimates for the magnetic Schrödinger equation, to appear on *Comm. Cont. Math.*

[12] Fanelli, L., and Vega, L., Magnetic virial identities, weak dispersion and Strichartz inequalities, to appear on *Math. Ann.*

[13] Georgiev, V., Stefanov, A., and Tarulli, M. Smoothing - Strichartz estimates for the Schrödinger equation with small magnetic potential, *Discrete Contin. Dyn. Syst.* A **17** (2007), 771–786.

[14] Ginibre, J., and Velo, G., Generalized Strichartz inequalities for the wave equation, *J. Funct. Anal.* **133** (1995) no. 1, 50–68.

[15] Goldberg, M., Dispersive estimates for the three-dimensional schrödinger equation with rough potential, *Amer. J. Math.* **128** (2006), 731–750.

[16] Goldberg, M., and Schlag, W., Dispersive estimates for schrödinger operators in dimensions one and three, *Comm. Math. Phys.* **251** (2004), 157–178.

[17] Goldberg, M., and Vega, L., and Visciglia, N., Counterexamples of Strichartz inequalities for Schrödinger equations with repulsive potentials, *Int. Math Res Not.*, 2006 Vol. 2006: article ID 13927.

[18] Ionescu, A.D., and Kenig, C., Well-posedness and local smoothing of solutions of Schrödinger equations, *Math. Res. Letters* **12** (2005), 193–205.

[19] Kato, T., and Yajima, K., Some examples of smooth operators and the associated smoothing effect, *Rev. Math. Phys.* **1** (1989), 481–496.

[20] Keel, M., and Tao, T., Endpoint Strichartz estimates, *Amer. J. Math.* **120** (1998) no. 5, 955–980.

[21] Leinfelder, H., and Simader, C., Schrödinger operators with singular magnetic vector potentials, *Math Z.* **176** (1981), 1–19.

[22] C.S. Morawetz, Time decay for the nonlinear Klein-Gordon equation, *Proc. Roy. Soc. London A*, **306** (1968), 291–296.

[23] B. Perthame, and L. Vega, Morrey-Campanato estimates for the Helmholtz Equations, *J. Func. Anal.* **164** (1999), 340–355.

[24] Robbiano, L., and Zuily, C., Strichartz estimates for Schrödinger equations with variable coefficients, *M*em. Soc. Math. Fr. **101-102** (2005).

[25] Rodnianski, I., and Schlag, W., Time decay for solutions of Schrödinger equations with rough and time-dependent potentials, *Invent. Math.* **155**(3) (2004), 451–513.

[26] Ruiz, A., and Vega, L., On local regularity of Schr¨odinger equations. *Int. Math. Research Notices* 1, 1993, 13–27 .

[27] Sjölin, P., Regularity of solutions to the Schrödinger equations, *Duke Math. J.* **55** (1987), 699–715.

[28] Stein, E., *Harmonic Analysis.* Princeton University Press, Princeton, New Jersey, 1993.

[29] Strichartz, R., Restriction of Fourier transforms to quadratic surfaces and decay of solutions of wave equations, *Duke Math. J.* **44** (1977), 705–774.

[30] Vega, L., The Schrödinger equation: pointwise convergence to the initial date, *Proc. AMS* **102** (1988), 874–878.

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