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The proof of the Nirenberg-Treves conjecture

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Abstract

We prove the Nirenberg-Treves conjecture: that for principal type pseudo-differential operators local solvability is equivalent to condition (Ψ). This condition rules out certain sign changes of the imaginary part of the principal symbol along the bicharacteristics of the real part. We obtain local solvability by proving a localizable estimate for the adjoint operator with a loss of two derivatives (compared with the elliptic case).

The proof involves a new metric in the Weyl (or Beals-Fefferman) calculus. This makes it possible to reduce to the case when the gradient of the imaginary part is non-vanishing, and then the zeroes form a smooth submanifold. The estimate uses a new type of weight, which measures the change of the distance to the zeroes of the imaginary part along the bicharacteristics of the real part between the minima of the curvature of this submanifold. By using condition (Ψ) and this weight, we can construct a multiplier which gives the estimate.

1. Introduction

We shall study the question of local solvability of a classical pseudo-differential operator $P \in \Psi^m_{cl}(M)$ on a C^∞ manifold M. Thus, we assume that the symbol of P is an asymptotic sum of homogeneous terms, and that $p = \sigma(P)$ is the homogeneous principal symbol of P. We shall also assume that P is of principal type, which means that the Hamilton vector field H_p and the radial vector field are linearly independent when $p = 0$.

Local solvability of P at a compact set $K \subseteq M$ means that the equation

$$Pu = v \tag{1.1}$$

has a local solution $u \in \mathcal{D}'(M)$ in a neighborhood of K for any $v \in C^\infty(M)$ in a set of finite codimension. We can also define microlocal solvability at any compactly based cone $K \subset T^*M$, see Definition 26.4.3 in [10].

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It was conjectured by Nirenberg and Treves [19] that condition (Ψ) was equivalent to local solvability of pseudo-differential operators of principal type. Condition (Ψ) means that

$$
\begin{aligned}
&\operatorname{Im}(ap) \text{ does not change sign from } - \text{ to } + \\
&\qquad\qquad \text{along the oriented bicharacteristics of } \operatorname{Re}(ap)
\end{aligned}
\tag{1.2}
$$

for any $0 \neq a \in C^\infty(T^*M)$; actually it suffices to check this for some $a \in C^\infty(T^*M)$ such that $H_{\operatorname{Re}(ap)} \neq 0$ by Theorem 26.4.12 in [10]. By oriented bicharacteristics of $\operatorname{Re}(ap)$ we mean the positive flow-out of the Hamilton vector field $H_{\operatorname{Re}(ap)} \neq 0$ on $\operatorname{Re}(ap) = 0$, these are also called semi-bicharacteristics. Condition (1.2) is invariant under conjugation with elliptic Fourier integral operators and multiplication with elliptic pseudo-differential operators, see Lemma 26.4.10 in [10].

For differential operators, condition (Ψ) is equivalent to condition (P), which rules out any sign changes of $\operatorname{Im}(ap)$ along the bicharacteristics of $\operatorname{Re}(ap)$ for $0 \neq a \in C^\infty(T^*M)$. The sufficiency of (P) for local solvability of principal type pseudo-differential operators was proved by Nirenberg and Treves [19] in the case when the principal symbol is real analytic, and by Beals and Fefferman [1] in the general case.

The necessity of (Ψ) for local solvability of principal type pseudo-differential operators was proved by Moyer in two dimensions and by Hörmander in general, see Corollary 26.4.8 in [10]. In the analytic category, the sufficiency of condition (Ψ) for solvability of principal type microdifferential operators acting on microfunctions was proved by Trépreau [20] (see also [11, Chapter VII]). The sufficiency of condition (Ψ) for local solvability of principal type pseudo-differential operators in two dimensions was proved by Lerner [13], leaving the higher dimensional case open.

Lerner [14] constructed counterexamples to the sufficiency of (Ψ) for local optimal (L^2) solvability of first order principal type pseudo-differential operators, raising doubts on whether the condition really was sufficient for solvability. But it was proved by the author [4] that Lerner's counterexamples are locally solvable with loss of at most two derivatives (compared with the elliptic case). Observe that optimal solvability of first order principal type pseudo-differential operators means a loss of one derivative. There are several results giving local solvability under conditions stronger than (Ψ), see [5], [12], [15] and [17].

In this paper we shall prove local solvability of principal type pseudo-differential operators $P \in \Psi^m_{cl}(M)$ satisfying condition (Ψ), this resolves the Nirenberg-Treves conjecture. To get local solvability we shall assume a strong form of the non-trapping condition at x_0: that all semi-characteristics are transversal to the fiber $T^*_{x_0}\mathbf{R}^n$, i.e., $p(x_0,\xi) = 0 \implies \partial_\xi p(x_0,\xi) \neq 0$.

Theorem 1.1. *If $P \in \Psi^m_{cl}(M)$ is of principal type satisfying condition (Ψ) near $x_0 \in M$ and $\partial_\xi p(x_0,\xi) \neq 0$ when $p(x_0,\xi) = 0$, then P is locally solvable at x_0.*

It follows from the proof that we lose at most two derivatives in the estimate of the adjoint, which is one more compared with the condition (P) case. Thus the result has the consequence that hypoelliptic operators of principal type can lose at most two derivatives. In fact, if the operator is hypoelliptic of principal type, then the adjoint is solvable of principal type, thus satisfying condition (Ψ) and we obtain an estimate of the operator.

Theorem 1.1 is going to be proved by the construction of a pseudo-sign which will be used in a multiplier estimate. The symbol of the pseudo-sign is, modulo elliptic factors, essentially a perturbation of the signed homogeneous distance to the sign changes of the imaginary part of the principal symbol.

Observe that Theorem 1.1 can be microlocalized: if condition (Ψ) holds microlocally near $(x_0, \xi_0) \in S^*(M)$ then P is microlocally solvable near (x_0, ξ_0), see Corollary 2.4. Since we lose two derivatives in the estimate this is not trivial, it is a consequence of the special type of estimate (see Remark 2.3). This paper is a shortened version of [7], we have excluded some of the longer and more technical proofs.

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2. Reduction to the multiplier estimate

In this section we shall reduce the proof of Theorem 1.1 to an estimate for a microlocal normal form of the adjoint for the operator. By using Darboux' theorem and the Malgrange Preparation Theorem, we may obtain the adjoint P^* on the following microlocal normal form

$$P_0 = D_t + iF(t, x, D_x) \tag{2.1}$$

where $F \in C(\mathbf{R}, \Psi^1_{cl}(T^*\mathbf{R}^n))$ has real principal symbol $\sigma(F) = f$. Observe that we do not assume that $t \mapsto f(t, x, \xi)$ is differentiable. Since P satisfies condition (Ψ) we find that P_0 satisfies condition $(\overline{\Psi})$:

$$t \mapsto f(t, x, \xi) \text{ does not change sign from } + \text{ to } - \text{ with increasing } t \text{ for any } (x, \xi). \tag{2.2}$$

We shall use the Weyl quantization of symbols $a(x, \xi) \in C^\infty(T^*\mathbf{R}^n) \bigcap \mathcal{S}'(T^*\mathbf{R}^n)$:

$$a^w(x, D_x)u(x) = (2\pi)^{-n} \iint \exp\left(i\langle x - y, \xi\rangle\right) a\left(\tfrac{x+y}{2}, \xi\right) u(y)\, dyd\xi \qquad u \in C_0^\infty(\mathbf{R}^n).$$

For Weyl calculus notations and results, see [10, Section 18.5]. Observe that $\operatorname{Re} a^w = (\operatorname{Re} a)^w$ is the symmetric part and $i \operatorname{Im} a^w = (i \operatorname{Im} a)^w$ the antisymmetric part of the operator a^w. Also, if $a \in S^m_{1,0}(T^*\mathbf{R}^n)$ then $a(x, D_x) \cong a^w(x, D_x)$ modulo $\Psi^{m-1}_{1,0}(T^*\mathbf{R}^n)$. In the following, we shall denote $S^m_{\varrho,\delta}(T^*\mathbf{R}^n)$ by $S^m_{\varrho,\delta}$, $0 \le \delta \le \varrho \le 1$.

Definition 2.1. We say that the symbol $b(x, \xi)$ is in $S^m_{1/2,1/2}$ of first order, if b satisfies the estimates in $S^m_{1/2,1/2}$ for derivatives of order ≥ 1.

This means that the homogeneous gradient $(\partial_x b, |\xi|\partial_\xi b) \in S^{m+\frac{1}{2}}_{1/2,1/2}$, and implies that the commutators of b^w with operators in $\Psi^k_{1,0}$ are in $\Psi^{m+k-1/2}_{1/2,1/2}$. Observe that this condition is preserved when multiplying with symbols in $S^0_{1,0}$.

We are going to prove an estimate for operators P_0 which satisfy condition (2.2). Let $\|u\|_{(s)}$ be the usual Sobolev norm, let $\|u\| = \|u\|_{(0)}$ be the L^2 norm, and $\langle u, v\rangle$ the corresponding inner product.

Proposition 2.2. *Assume that $P = D_t + iF^w(t,x,D_x)$, with $F \in C(\mathbf{R}, S^1_{cl})$ having real principal symbol f satisfying condition (2.2). Then there exists $T_0 > 0$ such that if $0 < T \le T_0$ then we can choose a real valued symbol $b_T(t,x,\xi) \in L^\infty(\mathbf{R}, S^{1/2}_{1/2,1/2})$ uniformly, with the property that $b_T \in S^0_{1/2,1/2}$ of first order uniformly, and*

$$\|u\|^2_{(-1/4)} \le T\,\mathrm{Im}\langle P_0 u, b^w_T u\rangle \tag{2.3}$$

for $u(t,x) \in C_0^\infty(\mathbf{R}\times\mathbf{R}^n)$ having support where $|t| \le T \le T_0$.

Note that we have to change the multiplier b_T when we change T, but that the multipliers are uniformly bounded in the symbol class. By the calculus, the conditions on b_T are preserved when composing b^w_T with symmetric operators in $L^\infty(\mathbf{R}, \Psi^0_{1,0})$.

Remark 2.3. *The estimate (2.3) can be perturbed with terms in $L^\infty(\mathbf{R}, S^0_{1,0})$ in the symbol of P_0 for small enough T. Thus it can be microlocalized: if $\phi(x,\xi) \in S^0_{1,0}$ is real valued then we have*

$$\mathrm{Im}\langle P_0\phi^w u, b^w_T\phi^w u\rangle \le \mathrm{Im}\langle P_0 u, \phi^w b^w_T \phi^w u\rangle + C\|u\|^2_{(-1/4)} \tag{2.4}$$

where $\phi^w b^w_T \phi^w$ satisfies the same conditions as b^w_T.

In fact, assume that $P_0 = D_t + if^w(t,x,D_x) + r^w(t,x,D_x)$ with $r \in L^\infty(\mathbf{R}, S^0_{1,0})$. By conjugation with $E^w(t,x,D_x)$ where

$$E(t,x,\xi) = \exp\left(-\int_0^t \mathrm{Im}\, r(s,x,\xi)\,ds\right) \in L^\infty(\mathbf{R}, S^0_{1,0}),$$

we can reduce to the case when $\mathrm{Im}\, r \in L^\infty(\mathbf{R}, S^{-1}_{1,0})$. We find that b^w_T is replaced with $B^w_T = E^w b^w_T E^w$, which is real and satisfies the same conditions as b^w_T since E is real. Clearly, the estimate (2.3) can be perturbed with terms in $L^\infty(\mathbf{R}, S^{-1}_{1,0})$ in the symbol expansion of P_0, and if $a(t,x,\xi) \in L^\infty(\mathbf{R}, S^0_{1,0})$ is real valued, then

$$\mathrm{Im}\langle a^w u, b^w_T u\rangle = \frac{1}{2i}\langle [b^w_T, a^w]u, u\rangle \le C\|u\|^2_{(-1/4)} \tag{2.5}$$

since $b_T \in S^0_{1/2,1/2}$ of first order, $\forall\, t$. We also find that $[P_0, \phi^w] \cong \{\, f, \phi \,\}^w$ modulo $L^\infty(\mathbf{R}, \Psi^{-1}_{1,0})$ where $\{\, f, \phi\,\} \in L^\infty\left(\mathbf{R}, S^0_{1,0}\right)$ is real valued. By using (2.5) with $a = \{\, f, \phi \,\}$, we obtain that the estimate (2.3) is localizable.

Proof of Theorem 1.1. By using Darboux' theorem and the Malgrange Preparation Theorem, we may assume that the adjoint P^* is equal to P_0 microlocally, where P_0 satisfies the conditions in Proposition 2.2 (see [10, Th. 21.3.6]). By using (2.3), a partition of unity, and the Cauchy-Schwarz inequality, we obtain $R \in S^0_{1,0}$, such that $x_0 \notin \mathrm{sing\,supp}\, R$ and

$$\|u\|_{(-1/4)} \le C\|P^* u\|_{(7/4-m)} + \|R^w u\|_{(-1/4)} \tag{2.6}$$

for $u(x) \in C_0^\infty(\mathbf{R}^n)$ having support where $|x| \le T_0$ is small enough. Now conjugation with $\langle D_x\rangle^s$ does not change the principal symbol of P. Thus, for any $s \in \mathbf{R}$ we may replace $-1/4$ by s and $7/4$ by $s+2$ in (2.6) after changing T_0 and R. This gives the local solvability of P with a loss of at most two derivatives, and finishes the proof of Theorem 1.1. □

Corollary 2.4. *If $P \in \Psi^m_{cl}(M)$ is of principal type near $(x_0, \xi_0) \in T^*M$, satisfying condition (Ψ) microlocally near (x_0, ξ_0), then P is microlocally solvable at (x_0, ξ_0).*

In order to prove Proposition 2.2 we shall need to make a "second microlocalization" using the specialized symbol classes of the Weyl calculus (see [10, Section 18.5]). Assume that $g_{x,\xi}(dx, d\xi)$ is a σ temperate metric on $T^*\mathbf{R}^n$, and let m be a σ, g temperate. Let $S(m, g)$ be the class of symbols $a \in C^\infty(T^*\mathbf{R}^n)$ with the seminorms

$$|a|^g_j(x,\xi) = \sup_{T_i \neq 0} \frac{|a^{(j)}(x,\xi,T_1,\dots,T_j)|}{\prod_1^j g_{x,\xi}(T_i)^{1/2}} \le C_j m(x,\xi) \qquad \forall\,(x,\xi) \text{ for } j \ge 0.$$

We shall use metrics which are conformal, they shall be on the form $g_{x,\xi}(dx, d\xi) = H(x,\xi)g^\sharp(dx, d\xi)$ where $0 < H(x,\xi) \le 1$ and $g^\sharp$ is a constant symplectic metric: $(g^\sharp)^\sigma = g^\sharp$. In the following, we say that $m > 0$ is a weight for a metric g if m is σ, g temperate.

Definition 2.5. Let m be a weight for the σ temperate metric g. We say that $a \in S^+(m, g)$ if $|a|^g_j \le C_j m$ for $j \ge 1$.

For example, $b \in S^+(1, g_{1/2,1/2})$, with $g_{1/2,1/2} = \langle\xi\rangle|dx|^2 + |d\xi|^2/\langle\xi\rangle$ at (x,ξ), if and only if $b \in S^0_{1/2,1/2}$ of first order. After microlocalizing where $\langle\xi\rangle \cong h^{-1} \ge 1$ is constant, and doing a microlocal change of coordinates, we find that $S^k_{1,0}$ corresponds to $S(h^{-k}, hg^\sharp)$ and $S^k_{1/2,1/2}$ corresponds to $S(h^{-k}, g^\sharp)$ microlocally. Thus we may reduce to the case in the following result (see the proof of Proposition 2.2 in [7]).

Proposition 2.6. *Assume that $P_0 = D_t + if^w(t, x, D_x)$, with real valued $f(t, x, \xi) \in C(\mathbf{R}, S(h^{-1}, hg^\sharp))$ satisfying condition (2.2), here $0 < h \le 1$ and $g^\sharp = (g^\sharp)^\sigma$ are constant. Then there exists $T_0 > 0$, such that if $0 < T \le T_0$ there exist a weight $h \le H_T \le 1$ for $g^\sharp$ and a real valued symbol $b_T(t, x, \xi) \in L^\infty(\mathbf{R}, S(H_T^{-1/2}, g^\sharp) \bigcap S^+(1, g^\sharp))$ uniformly, so that*

$$h^{1/2} \int \|u\|^2(t)\,dt \le C_0 T \int \operatorname{Im}\langle Pu, b^w_T u\rangle(t)\,dt \tag{2.7}$$

for $u(t, x) \in C^\infty_0(\mathbf{R} \times \mathbf{R}^n)$ having support where $|t| \le T \le T_0$.

The conditions on b_T means in $g^\sharp$ orthonormal coordinates that $|b_T| \le CH_T^{-1/2}$ and $|\partial^\alpha_x \partial^\beta_\xi b_T| \le C_{\alpha\beta}$ when $|\alpha| + |\beta| \ge 1$. As before, the estimate (2.7) can be perturbed with terms in $L^\infty(\mathbf{R}, S(1, hg^\sharp))$ in the symbol of P_0 for small T (with changed b_T), and it can be localized with respect to the metric $hg^\sharp$. Next, we shall state and prove the multiplier estimate that we are going to use for the proof of Proposition 2.6.

Let $\mathcal{B} = \mathcal{B}(L^2(\mathbf{R}^n))$ be the set of bounded operators $L^2(\mathbf{R}^n) \mapsto L^2(\mathbf{R}^n)$. We say that $A(t) \in C(\mathbf{R}, \mathcal{B})$ if $A(t) \in \mathcal{B}$ for all $t \in \mathbf{R}$ and $t \mapsto A(t)u \in C(\mathbf{R}, L^2(\mathbf{R}^n))$ for any $u \in L^2(\mathbf{R}^n)$. We shall consider the operator

$$P = D_t + iF(t) \tag{2.8}$$

where $F(t) \in C(\mathbf{R}, \mathcal{B})$. In the applications, we will have $F(t) \in C(\mathbf{R}, \operatorname{Op} S(h^{-1}, hg^\sharp))$ where h is constant. But we shall also use multipliers which are not continuous in t. In the following, we let $\|u\|(t)$ be the L^2 norm of $u(t,x)$ in $\mathbf{R}^n$ for fixed t, and $\langle u, v\rangle(t)$ the corresponding inner product.

Definition 2.7. We say that $A(t)$ is in $L^\infty_{loc}(\mathbf{R}, \mathcal{B})$ if $A(t) \in \mathcal{B}$ for any t, and $t \mapsto A(t)u$ is in $L^\infty_{loc}(\mathbf{R}, L^2(\mathbf{R}^n))$ for any $u \in L^2(\mathbf{R}^n)$, i.e., $t \mapsto \langle A(t)u, v\rangle$ is in $L^\infty_{loc}(\mathbf{R})$ for any $u, v \in L^2(\mathbf{R}^n)$.

If $A(t) \in L^\infty_{loc}(\mathbf{R}, \mathcal{B})$, then we find that $t \mapsto \langle A(t)u, u\rangle \in L^\infty_{loc}(\mathbf{R})$ has weak derivative $\langle \frac{d}{dt} A(\cdot)u, u\rangle \in \mathcal{D}'(\mathbf{R})$ for any $u \in \mathcal{S}(\mathbf{R}^n)$ given by

$$\langle \tfrac{d}{dt} A(\cdot)u, u\rangle(\phi) = -\int \langle A(t)u, u\rangle \phi'(t)\, dt, \qquad \phi(t) \in C_0^\infty(\mathbf{R}).$$

It is also easy to see that if $u(t), v(t) \in C(\mathbf{R}, L^2(\mathbf{R}^n))$ and $A(t) \in L^\infty_{loc}(\mathbf{R}, \mathcal{B})$, then $t \mapsto \langle A(t)u(t), v(t)\rangle \in L^\infty_{loc}(\mathbf{R})$.

We shall use the following multiplier estimate (see also [13] and [15] for similar estimates).

Proposition 2.8. *Let $P = D_t + iF(t)$ with $F(t) \in C(\mathbf{R}, \mathcal{B})$. Assume that $B(t) = B^*(t) \in L^\infty_{loc}(\mathbf{R}, \mathcal{B})$ satisfies*

$$\operatorname{Re}\langle \tfrac{d}{dt} B(t)u, u\rangle + 2\operatorname{Re}\langle B(t)u, F(t)u\rangle \ge \operatorname{Re}\langle m(t)u, u\rangle \quad \text{in } \mathcal{D}'(I) \qquad \forall\, u \in C_0^\infty(\mathbf{R}^n) \tag{2.9}$$

where $m(t) \in L^\infty_{loc}(\mathbf{R}, \mathcal{B})$ and $I \subseteq \mathbf{R}$ is an open interval. Then we have

$$\int \operatorname{Re}\langle m(t)u(t), u(t)\rangle\, dt \le 2\int \operatorname{Im}\langle Pu(t), B(t)u(t)\rangle\, dt \tag{2.10}$$

for any $u \in C_0^1(I, C_0^\infty(\mathbf{R}^n))$.

Proof. Since $B(t) \in \mathcal{B}$ is weakly measurable and locally bounded, we may for $u \in C_0^\infty(\mathbf{R}^n)$ define the regularization

$$\langle B_\varepsilon(t)u, u\rangle = \varepsilon^{-1}\int \langle B(s)u, u\rangle \phi((t-s)/\varepsilon)\, ds = \langle Bu, u\rangle(\phi_{\varepsilon,t}) \qquad \varepsilon > 0$$

where $\phi_{\varepsilon,r}(s) = \varepsilon^{-1}\phi((r-s)/\varepsilon)$ with $0 \le \phi \in C_0^\infty(\mathbf{R})$ satisfying $\int \phi(t)\, dt = 1$. Then $t \mapsto \langle B_\varepsilon(t)u, u\rangle$ is in $C^\infty(\mathbf{R})$ with derivative at $t = r$ equal to $\langle \frac{d}{dt} B_\varepsilon(r)u, u\rangle = \frac{d}{dt}\langle Bu, u\rangle(\phi_{\varepsilon,r})$. Let I_0 be an open interval such that $I_0 \Subset I$. Then for small enough $\varepsilon > 0$ we find from condition (2.9) that

$$\operatorname{Re}\langle \tfrac{d}{dt} B_\varepsilon(t)u, u\rangle + 2\operatorname{Re}\langle Bu, Fu\rangle(\phi_{\varepsilon,t}) \ge \operatorname{Re}\langle mu, u\rangle(\phi_{\varepsilon,t}) \qquad t \in I_0 \quad u \in C_0^\infty(\mathbf{R}^n). \tag{2.11}$$

In fact, $\phi_{\varepsilon,t} \ge 0$ and $\operatorname{supp}\phi_{\varepsilon,t} \in C_0^\infty(I)$ for small enough ε when $t \in I_0$.

Now we define for $u \in C_0^1(I_0, C_0^\infty(\mathbf{R}^n))$ and small enough $\varepsilon > 0$

$$M_{\varepsilon,u}(t) = \operatorname{Re}\langle B_\varepsilon u, u\rangle(t) = \varepsilon^{-1}\int \langle B(s)u(t), u(t)\rangle \phi((t-s)/\varepsilon)\, ds. \tag{2.12}$$

By differentiating under the integral sign we obtain that $M_{\varepsilon,u}(t) \in C_0^1(I_0)$, with derivative $\frac{d}{dt}M_{\varepsilon,u} = \mathrm{Re}\langle(\frac{d}{dt}B_\varepsilon)u, u\rangle + 2\,\mathrm{Re}\langle B_\varepsilon u, \partial_t u\rangle$ since $B(t) \in L^\infty_{loc}(\mathbf{R}, \mathcal{B})$. By integrating with respect to t, we obtain the vanishing average

$$0 = \int M_{\varepsilon,u}(t)\,dt = \int \mathrm{Re}\langle(\tfrac{d}{dt}B_\varepsilon)u, u\rangle\,dt + \int 2\,\mathrm{Re}\langle B_\varepsilon u, \partial_t u\rangle\,dt \tag{2.13}$$

when $u \in C_0^1(I_0, C_0^\infty(\mathbf{R}^n))$. Since $\partial_t u = iPu + Fu$ we obtain from (2.11) and (2.13) that

$$0 \ge \iint \big(\mathrm{Re}\langle m(s)u(t), u(t)\rangle + 2\,\mathrm{Re}\langle B(s)u(t), iPu(t)\rangle \\ + \mathrm{Re}\langle B(s)u(t), \big(F(t) - F(s)\big)u(t)\rangle\big)\phi_{\varepsilon,t}(s)\,dsdt.$$

By letting $\varepsilon \to 0$ we obtain by dominated convergence that

$$0 \ge \int \mathrm{Re}\langle m(t)u(t), u(t)\rangle + 2\,\mathrm{Re}\langle B(t)u(t), iPu(t)\rangle\,dt$$

since $F(t) \in C(\mathbf{R}, \mathcal{B})$, $u \in C_0^1(I_0, C_0^\infty(\mathbf{R}^n))$, $m(t)$ and $B(t)$ are uniformly bounded in $\mathcal{B}$ when $t \in \operatorname{supp} u$. Now $2\,\mathrm{Re}\langle Bu, iPu\rangle = -2\,\mathrm{Im}\langle Pu, Bu\rangle$, thus we obtain (2.10) for $u \in C_0^1(I_0, C_0^\infty(\mathbf{R}^n))$. Since I_0 is an arbitrary open subinterval with compact closure in I, this completes the proof of the proposition. □

Now we can reduce the proof of Proposition 2.6 to the construction of a pseudo-sign $B = b^w$ in a fixed interval.

Proposition 2.9. *Assume that $f \in C(\mathbf{R}, S(h^{-1}, hg^\sharp))$ is a real valued symbol satisfying condition $(\overline{\Psi})$ given by (2.2), here $0 < h \le 1$ and $g^\sharp = (g^\sharp)^\sigma$ are constant. Then there exist a positive constant c_0, a weight $h \le H_1 \le 1$ for $g^\sharp$, real valued symbols $b(t, x, \xi) \in L^\infty(\mathbf{R}, S(H_1^{-1/2}, g^\sharp) + S^+(1, g^\sharp))$ and $\mu(t, x, \xi) \in L^\infty(\mathbf{R}, S(1, g^\sharp))$ such that for any $u(x) \in C_0^\infty(\mathbf{R}^n)$ we have*

$$\begin{cases} \langle \partial_t(b^w)u, u\rangle \ge \langle \mu^w u, u\rangle \ge c_0 h^{1/2}\|u\|^2 \\ \mathrm{Re}\langle b^w f^w u, u\rangle \ge -\langle \mu^w u, u\rangle / c_0 \end{cases} \quad \text{in } \mathcal{D}'(\mathbf{R}) \text{ when } |t| < 1.$$

Here c_0, and the seminorms of b and m only depend on the seminorms of f in $S(h^{-1}, hg^\sharp)$ for $|t| \le 1$.

Proof of Proposition 2.6. By doing a dilation $s = t/T$, we find that P transforms into $T^{-1}P_T = T^{-1}(D_s + iTf_T^w(s, x, D_x))$, where $f_T(s, x, \xi) = f(Ts, x, \xi)$ satisfies the conditions in Proposition 2.9 uniformly in T when $0 < T \le 1$. Thus we obtain real b_T, μ_T and c_0 such that when $|s| < 1$ we have

$$\begin{cases} \langle \partial_s(b_T^w)u, u\rangle \ge \langle \mu_T^w u, u\rangle \ge c_0 h^{1/2}\|u\|^2 \\ \mathrm{Re}\langle b_T^w f_T^w u, u\rangle \ge -\langle \mu_T^w u, u\rangle / c_0 \end{cases} \quad \text{in } \mathcal{D}'(\mathbf{R})$$

for $u \in C_0^\infty(\mathbf{R}^n)$. This implies that

$$\langle \partial_s b_T^w(s, x, D_x)u, u\rangle + 2\,\mathrm{Re}\langle Tf_T^w(s, x, D_x)u, b_T^w(s, x, D_x)u\rangle \\ \ge (1 - 2T/c_0)\langle \mu_T^w(s, x, D_x)u, u\rangle \qquad \text{in } \mathcal{D}'\big(]{-1}, 1[\big)$$

for $u \in C_0^\infty(\mathbf{R}^n)$. Thus, for $T \le c_0/4$ we obtain by using Proposition 2.8 with $P_T = D_s + i\,T f_T^w(s,x,D_x)$, $B(s) = b_T^w(s,x,D_x)$ and $m(s) = \mu_T^w(s,x,D_x)$ that

$$c_0 h^{1/2} \int \|u\|^2\, ds \le \int \langle \mu_T^w u, u\rangle\, ds \le 4 \int \mathrm{Im}\langle P_T u, b_T^w u\rangle(s)\, ds$$

if $u \in C_0^\infty(\mathbf{R}\times\mathbf{R}^n)$ has support where $|s| < 1$. Finally, we obtain that

$$c_0 h^{1/2} \int \|u\|^2\, dt \le 4T \int \mathrm{Im}\langle Pu, \widetilde{b}_T^w u\rangle(t)\, dt$$

with $\widetilde{b}_T(t,x,\xi) = b_T(t/T,x,\xi)$ for $u \in C_0^\infty(\mathbf{R}\times\mathbf{R}^n)$ has support where $|t| < T \le c_0/4$. □

It remains to prove Proposition 2.9, which will be done in Section 5.

3. Symbol Classes and Weights

Next, we shall define the symbol classes we shall use. In the following, we shall denote (x,ξ) by $w \in T^*\mathbf{R}^n$, and we shall assume that $f \in C(\mathbf{R}, S(h^{-1}, hg^\sharp))$ satisfies condition $(\overline{\Psi})$ given by (2.2), here $0 < h \le 1$ and $g^\sharp = (g^\sharp)^\sigma$ are constant. We shall only consider the values of $f(t,w)$ when $|t| \le 1$, thus for simplicity we let $f(t,w) = f(1,w)$ when $t \ge 1$ and $f(t,w) = f(-1,w)$ when $t \le -1$. In the following, the results will be uniform in the sense that they will only depend on the seminorms of f in $S(h^{-1}, hg^\sharp)$.

First, we shall define the signed distance function $\delta_0(t,w)$ in $T^*\mathbf{R}^n$ for fixed $t \in \mathbf{R}$, with the property that $t \mapsto \delta_0(t,w)$ is non-decreasing and $\delta_0 f \ge 0$. Let

$$X_+ = \{\, (t,w) \in \mathbf{R}\times T^*\mathbf{R}^n : \exists\, s \le t,\ f(s,w) > 0 \,\} \tag{3.1}$$
$$X_- = \{\, (t,w) \in \mathbf{R}\times T^*\mathbf{R}^n : \exists\, s \ge t,\ f(s,w) < 0 \,\}. \tag{3.2}$$

We have that $X_\pm$ are open in $\mathbf{R}\times T^*\mathbf{R}^n$, and by condition $(\overline{\Psi})$ we obtain that $X_- \bigcap X_+ = \emptyset$ and $\pm f \ge 0$ on $X_\pm$. Let $X_0 = \mathbf{R}\times T^*\mathbf{R}^n \setminus (X_+ \bigcup X_-)$, which is closed in $\mathbf{R}\times T^*\mathbf{R}^n$, by the definition of $X_\pm$ we have $f = 0$ on X_0. Let

$$d_0(t_0,w_0) = \inf\{\, g^\sharp(w_0 - z)^{1/2} :\ (t_0,z) \in X_0 \,\}$$

which is the $g^\sharp$ distance in $T^*\mathbf{R}^n$ to X_0 for fixed t_0, it is identically equal to $+\infty$ in the case that $X_0 \bigcap \{\, t = t_0 \,\} = \emptyset$.

Definition 3.1. We say that $w \mapsto a(w)$ is Lipschitz continuous on $T^*\mathbf{R}^n$ with respect to the metric $g^\sharp$ if

$$\sup_{w\ne z \in T^*\mathbf{R}^n} |a(w)-a(z)|/g^\sharp(w-z)^{1/2} = C < \infty$$

and then C is the Lipschitz constant of a. We shall denote by $\mathrm{Lip}(T^*\mathbf{R}^n)$ the Lipschitz continuous functions on $T^*\mathbf{R}^n$ with respect to the metric $g^\sharp$.

By using the triangle inequality and taking the infimum over z we find that $w \mapsto d_0(t, w)$ is Lipschitz continuous with respect to the metric $g^\sharp$ with Lipschitz constant equal to 1, for those t when it is not identically equal to ∞.

Definition 3.2. We define the sign of f by $\operatorname{sgn}(f) = \pm 1$ on $X_\pm$ and $\operatorname{sgn}(f) = 0$ on X_0, then $t \to \operatorname{sgn}(f)(t, w)$ is non-decreasing and $\operatorname{sgn}(f) \cdot f \geq 0$. We define the signed distance function δ_0 by

$$\delta_0(t, w) = \operatorname{sgn}(f)(t, w) \min(d_0(t, w), h^{-1/2}). \tag{3.3}$$

By the definition we have that $|\delta_0| \leq h^{-1/2}$ and $|\delta_0| = d_0$ when $|\delta_0| < h^{-1/2}$. The signed distance function has the following properties.

Remark 3.3. *The signed distance function $w \mapsto \delta_0(t, w)$ given by Definition 3.2 is Lipschitz continuous with respect to the metric $g^\sharp$ with Lipschitz constant equal to 1. We also find that $\delta_0(t, w) f(t, w) \geq 0$ and $t \mapsto \delta_0(t, w)$ is non-decreasing.*

In fact, since $\delta_0 = 0$ on X_0 it suffices to show the Lipschitz continuity of $w \mapsto \delta_0(t, w)$ on X_+ and X_-, which follows from the Lipschitz continuity of $w \mapsto d_0(t, w)$. Since $(t, w) \in X_+$ implies $(s, w) \in X_+$ for $s \geq t$ and $(t, w) \in X_-$ implies $(s, w) \in X_-$ for $s \leq t$, it is easy to see that $t \mapsto \delta_0(t, w)$ is non-decreasing. Since $t \mapsto \delta_0(t, w)$ is non-decreasing and bounded, it is a regulated function. This means that the left and right limits $\delta_0(t\pm, w) = \lim_{0<\varepsilon\to 0} \delta_0(t \pm \varepsilon, w)$ exist for any (t, w) (see [8]).

In the following, we shall omit the parameter t, and denote $f' = \partial_w f$ and $f'' = f^{(2)}$, where the differentiation is in the w variables only. We shall also in the following assume that we have choosen $g^\sharp$ orthonormal coordinates so that $g^\sharp(dw) = |dw|^2$. We shall use the norms $|f'|_{g^\sharp} = |f'|$ and $\|f''\|_{g^\sharp} = \|f''\|$, but omit the index $g^\sharp$.

Definition 3.4. Let

$$H_1^{-1/2} = 1 + |\delta_0| + \frac{|f'|}{\|f''\| + h^{1/4}|f'|^{1/2} + h^{1/2}} \tag{3.4}$$

and $G_1 = H_1 g^\sharp$ the corresponding metric.

Since $|f'|/(\|f''\| + h^{1/4}|f'|^{1/2} + h^{1/2})$ is continuous in (t, w) we find that $t \mapsto H_1^{1/2}(t, w)$ is a regulated function. We also have

$$1 \leq H_1^{-1/2} \leq 1 + |\delta_0| + h^{-1/4}|f'|^{1/2} \leq Ch^{-1/2} \tag{3.5}$$

since $|f'| \leq C_1 h^{-1/2}$ and $|\delta_0| \leq h^{-1/2}$. Moreover, $|f'| \leq H_1^{-1/2}(\|f''\| + h^{1/4}|f'|^{1/2} + h^{1/2})$ so the Cauchy-Schwarz inequality gives

$$|f'| \leq 2\|f''\| H_1^{-1/2} + 3h^{1/2} H_1^{-1} \leq C H_1^{-1/2}. \tag{3.6}$$

In the case $1 + |\delta_0(w_0)| \leq H_1^{-1/2}(w_0)/2$ we have

$$H_1^{-1/2}(w_0) \leq 2|f'(w_0)|/(\|f''(w_0)\| + h^{1/4}|f'(w_0)|^{1/2} + h^{1/2}), \tag{3.7}$$

then we find

$$\|f''(w_0)\| \le 2H_1^{1/2}(w_0)|f'(w_0)| \qquad \text{and} \tag{3.8}$$

$$h^{1/2} \le 4H_1(w_0)|f'(w_0)|. \tag{3.9}$$

By Proposition 3.6 below the metric G_1 is σ temperate. The denominator

$$D = \|f''\| + h^{1/4}|f'|^{1/2} + h^{1/2} \tag{3.10}$$

in (3.4) may seem strange, but it has the following natural explanation which we owe to Nicolas Lerner [18]. We have $F = h^{-1/2}f \in S(h^{-3/2}, g)$, and the largest $H_2 \le 1$ for which $F \in S(H_2^{-3/2}, H_2 g^\sharp)$ is given by

$$H_2^{-1/2} \cong 1 + |F|^{1/3} + |F'|^{1/2} + \|F''\| = 1 + h^{-1/6}|f|^{1/3} + h^{-1/4}|f'|^{1/2} + h^{-1/2}\|f''\|$$

modulo bounded factors. We obtain that $H_2^{-1/2} \cong 1 + h^{-1/4}|f'|^{1/2} + h^{-1/2}\|f''\| = Dh^{-1/2}$ and $H_1^{-1/2} \cong 1 + |\delta_0| + |F'|H_2^{1/2} \le CH_2^{-1/2}$ in a G_2 neighborhood of $f^{-1}(0)$, and $H_2^{-1/2} \cong H_1^{-1/2} + \|F''\|$ in a G_1 neighborhood of $f^{-1}(0)$.

Definition 3.5. Let

$$M = |f| + |f'|H_1^{-1/2} + \|f''\|H_1^{-1} + h^{1/2}H_1^{-3/2}. \tag{3.11}$$

Then we have $h^{1/2} \le M \le ch^{-1}$, and M has the following properties.

Proposition 3.6. *We find that G_1 is σ temperate such that $G_1 = H_1^2 G_1^\sigma$ and*

$$H_1(w) \le C_0 H_1(w_0)(1 + H_1(w)g^\sharp(w - w_0)) \le C_0 H_1(w_0)(1 + g^\sharp(w - w_0)). \tag{3.12}$$

We also have that M is a weight for G_1 such that

$$M(w) \le C_1 M(w_0)(1 + H_1(w_0)g^\sharp(w - w_0))^{3/2} \le C_1 M(w_0)(1 + g^\sharp(w - w_0))^{3/2} \tag{3.13}$$

and $f \in S(M, G_1)$.

Observe that H_T is a weight for $g^\sharp$ since $G_T \le g^\sharp$. The advantage of using the metric G_1 is that in the case $H_1 \ll 1$ in a G_1 neighborhood of the sign changes, we obtain that $|f'| \ge ch^{1/2}$ is a weight for G_1, $\delta_0 \in S(H_1^{-1/2}, G_1)$ and the curvature of $f^{-1}(0)$ is bounded by $CH_1^{1/2}$ (see Remark 3.7 and Proposition 3.8 below).

Proof of Proposition 3.6. Now, if $G_{1,w_0}(w - w_0) \ge c$ then $g^\sharp(w - w_0) = |w - w_0|^2 \ge cH_1^{-1}(w_0)$ which immediately gives (3.12). Thus it suffices to show that G_1 is slowly varying in order to prove (3.12).

First we consider the case $1 + d_0(w_0) \ge H_1^{-1/2}(w_0)/2$. Then we find by the uniform Lipschitz continuity of $w \mapsto d_0(w)$ that

$$H_1^{-1/2}(w) \ge 1 + d_0(w) \ge 1 + d_0(w_0) - H_1^{-1/2}(w_0)/6 \ge H_1^{-1/2}(w_0)/3$$

when $|w - w_0| \le H_1^{-1/2}(w_0)/6$, which gives slow variation in this case.

In the case $1+d_0(w_0) \le H_1^{-1/2}(w_0)/2$ we obtain from Taylor's formula and (3.8) that

$$|f'(w)| \le |f'(w_0)| + \varepsilon H_1^{-1/2}(w_0)\|f''(w_0)\| + C\varepsilon^2 h^{1/2} H_1^{-1}(w_0)$$
$$\le (1+2\varepsilon+4C\varepsilon^2)|f'(w_0)| \qquad \text{when } |w-w_0| \le \varepsilon H_1^{-1/2}(w_0)$$

and similarly $|f'(w)| \ge (1-2\varepsilon-4C\varepsilon^2)|f'(w_0)|$. Thus we obtain that

$$1 - C'\varepsilon \le |f'(w)|/|f'(w_0)| \le 1 + C'\varepsilon \qquad \text{when } |w-w_0| \le \varepsilon H_1^{-1/2}(w_0) \tag{3.14}$$

in the case $H_1^{1/2} \le 1/4$ and $|\delta_0| \le H_1^{-1/2}/4$ at w_0. Taylor's formula and (3.9) gives

$$\|f''(w)\| \le \|f''(w_0)\| + C\varepsilon H_1^{-1/2}(w_0)h^{1/2} \le \|f''(w_0)\| + 4C\varepsilon H_1^{1/2}(w_0)|f'(w_0)|$$

when $|w-w_0| \le \varepsilon H_1^{-1/2}(w_0)$. Thus we obtain from (3.7) and (3.14) that

$$H_1^{1/2}(w) \le \|f''(w)\||f'|^{-1}(w) + h^{1/4}|f'|^{-1/2}(w) + h^{1/2}|f'|^{-1}(w) \le 3H_1^{1/2}(w_0)$$

when $|w-w_0| \le \varepsilon H_1^{-1/2}(w_0)$ and ε is small enough, which gives the slow variation.

Next, we prove (3.13). Taylor's formula gives as before that

$$\|f^{(k)}(w)\| \le C\left(\sum_{j=0}^{2-k} \|f^{(k+j)}(w_0)\||w-w_0|^j + h^{1/2}|w-w_0|^{3-k}\right) \qquad 0 \le k \le 2. \tag{3.15}$$

Thus we obtain from Definition 3.5 that

$$M(w) \le C\sum_{k=0}^{2} \|f^{(k)}(w_0)\|(|w-w_0| + H_1^{-1/2}(w))^k + Ch^{1/2}(|w-w_0| + H_1^{-1/2}(w))^3.$$

We obtain from (3.12) that $H_1^{-1/2}(w) \le C(H_1^{-1/2}(w_0) + |w-w_0|)$. This gives

$$M(w) \le C\sum_{k=0}^{2} \|f^{(k)}(w_0)\|H_1^{-k/2}(w_0)(1+H_1^{1/2}(w_0)|w-w_0|)^k$$
$$+ Ch^{1/2}H_1^{-3/2}(w_0)(1+H_1^{1/2}(w_0)|w-w_0|)^3 \le C'M(w_0)(1+H_1^{1/2}(w_0)|w-w_0|)^3$$

and (3.13).

It is clear from the definition of M that $\|f^{(k)}\| \le MH_1^{k/2}$ when $k \le 2$, and when $k \ge 3$ we have

$$\|f^{(k)}\| \le C_k h^{\frac{k-2}{2}} \le C_k' h^{1/2} H_1^{\frac{k-3}{2}} \le C_k' M H_1^{\frac{k}{2}}$$

since $h \le CH_1$ by (3.5) and $h^{1/2}H_1^{-3/2} \le M$. This completes the proof. □

Observe that $f \in S(M, H_1 g^\sharp)$ for any choice of $H_1 \ge ch$ in Definition 3.5, we do not use any other property of H_1.

Remark 3.7. *When* $1+|\delta_0(w_0)| \le H_1^{-1/2}(w_0)/2$ *we find that* $|f'(w_0)| \ge h^{1/2}/4$ *and*

$$1/C \le |f'(w)|/|f'(w_0| \le C \qquad \text{for } |w-w_0| \le \varepsilon H_1^{-1/2}(w_0). \tag{3.16}$$

We also have that $f' \in S(|f'|, G_1)$*, i.e.,*

$$|f^{(k)}(w_0)| \le C_k |f'(w_0)| H_1^{\frac{k-1}{2}}(w_0) \qquad \text{for } k \ge 1 \tag{3.17}$$

when $1+|\delta_0(w_0)| \le H_1^{-1/2}(w_0)/2$*.*

In fact, (3.17) is trivial if $k=1$, follows from (3.8) for $k=2$, and when $k \ge 3$ we have

$$|f^{(k)}(w_0)| \le C_k h^{\frac{k-2}{2}} \le 4C_k |f'| H_1 h^{\frac{k-3}{2}} \le C_k' |f'| H_1^{\frac{k-1}{2}}$$

by (3.5) and (3.9).

Proposition 3.8. *Let* $H_1^{-1/2}$ *be given by Definition 3.4 for* $f \in S(h^{-1}, hg^\sharp)$*. There exists* $\kappa_1 > 0$ *so that if* $|\delta_0(w_0)| \le \kappa_1 H_1^{-1/2}(w_0)$*,* $H_1^{1/2}(w_0) \le \kappa_1$ *and*

$$\partial_{w_1} f(w_0) \ge \kappa_1 |f'(w_0)| \tag{3.18}$$

then there exists $c_1 > 0$ *such that*

$$f(w) = \alpha_1(w)(w_1 - \beta(w')) \tag{3.19}$$

$$\delta_0(w) = \alpha_0(w)(w_1 - \beta(w')) \tag{3.20}$$

when $|w-w_0| \le c_1 H_1^{-1/2}(w_0)$*. Here* $0 < c_1 \le \alpha_0 \in S(1, G_1)$*,* $c_1|f'| \le \alpha_1 \in S(|f'|, G_1)$ *and* $\beta \in S(H_1^{-1/2}, G_1)$ *only depends on* w'*,* $w=(w_1,w')$*.*

Proof. We shall choose coordinates so that $w_0 = 0$, and put $H_1 = H_1(0)$. Since $H_1^{1/2} \le \kappa_1$ and $\delta_0(0) = 0$ we find from (3.7) and the slow variation that

$$\|f''(w)\||f'(w)|^{-1} + h^{1/4}|f'(w)|^{-1/2} \le C H_1^{1/2} \tag{3.21}$$

when $|w| \le \varepsilon H_1^{-1/2}$ for sufficiently small ε and κ_1. Remark 3.7 gives that $|f'(w)| \le C|f'(0)|$ when $|w| \le \varepsilon H_1^{-1/2}$ for small ε. We find from (3.21) that

$$\partial_{w_1} f(w) \ge \partial_{w_1} f(0) - C'\varepsilon |f'(0)| \ge \kappa_1 |f'(0)|/2 \ge ch^{1/2}$$

when $|w| \le \varepsilon H_1^{-1/2}$ and $\varepsilon > 0$ is small enough. Thus, by the implicit function theorem we can solve

$$f(w) = 0 \iff w_1 = \beta(w') \qquad \text{when } |w| \le \varepsilon H_1^{-1/2}$$

for sufficiently small $\varepsilon > 0$. We find that $\beta(0) = 0$, $|\beta'| = |\partial_{w'} f|/|\partial_{w_1} f| = \mathcal{O}(1)$ and

$$|\beta''| \le C(|\partial_{w_1}^2 f||\beta'|^2 + 2|\partial_{w'}\partial_{w_1} f||d\beta| + \|\partial_{w'}^2 f\|)/|\partial_{w_1} f| = \mathcal{O}(H_1^{1/2})$$

when $w_1 = \beta(w')$ and $|w| \le \varepsilon H_1^{-1/2}$.

Assume by induction that $|\partial^\alpha\beta| \le C_k H_1^{(|\alpha|-1)/2}$ when $|w| \le \varepsilon H_1^{-1/2}(0)$ for $|\alpha| < N$, where $N \ge 3$. Then we find for $|\alpha| = N$

$$\partial^\alpha\beta = -\big(\sum c_\gamma \partial_{w_1}^k \partial_{w'}^{\gamma_0} f \prod_{j=1}^k \partial_{w'}^{\gamma_j}\beta + \partial_{w'}^\alpha f\big)/\partial_{w_1} f \quad \text{when } w_1 = \beta(w') \text{ and } |w| \le \varepsilon H_1^{-1/2}$$

where the sum is over $k \ge 2$ and $\sum_{j=1}^k \gamma_j + \gamma_0 = \alpha$; or $k = 1$, $\gamma_0 \ne 0$ and $\gamma_0 + \gamma_1 = \alpha$. In any case, we obtain that $k + |\gamma_0| \ge 2$.

When $|\alpha| = N \ge 3$ we find $|\partial^\alpha f/\partial_{w_1} f| \le C_N h^{(|\alpha|-2)/2}/|\partial_{w_1} f| \le C_N' H_1^{(|\alpha|-1)/2}$, since we have $h \le CH_1$ by (3.5) and $h^{1/2}/|\partial_{w_1} f| \le CH_1$ by (3.21) when $|w| \le \varepsilon H_1^{-1/2}$. Similarly, for $k + |\gamma_0| \ge 2$ we find by the induction hypothesis that

$$\Big|\partial_{w_1}^k \partial_{w'}^{\gamma_0} f \prod_{j=1}^k \partial_{w'}^{\gamma_j}\beta/\partial_{w_1} f\Big| \le C_N H_1^{(|\alpha|-1)/2}$$

when $|w| \le \varepsilon H_1^{-1/2}$. In fact, the case $k + |\gamma_0| \ge 3$ works as before since $\sum_{j=0}^k |\gamma_j| = |\alpha|$, and when $k + |\gamma_0| = 2$ we use that $\|f''(w)\|/|\partial_{w_1} f(w)| \le CH_1^{1/2}$. This completes the induction argument.

Now by using Taylor's formula we find $f(w) = \alpha(w)(w_1 - \beta(w'))$ where

$$\alpha(w) = \int_0^1 \partial_{w_1} f(\theta w_1 + (1-\theta)\beta(w'), w')\, d\theta \qquad |w| \le \varepsilon H_1^{-1/2}.$$

Thus $\alpha(w) \cong |f'(0)|$ since $|\beta(w')| \le C\varepsilon H_1^{-1/2}$ when $|w| \le \varepsilon H_1^{-1/2}$. Now we have $\partial_{w_1} f \in S(|df|, G_1)$ by Remark 3.7, so $\alpha(w) = f_0(w, \beta(w'))$ for some $f_0 \in S(|df|, G_1)$ when $|w| \le \varepsilon H_1^{-1/2}$. Thus differentiation gives

$$|\partial^\gamma \alpha| \le C \sum_{\sum_{j=1}^k \gamma_j + \gamma_0 = \gamma} \Big|\partial_{w_1}^k \partial_{w'}^{\gamma_0} f_0 \prod_{j=1}^k \partial^{\gamma_j}\beta\Big| \le C'|df| H_1^{|\gamma|/2}$$

which proves (3.19).

It remains to prove the statements about $\delta_0(w)$. Let $G_{1,0} = H_1 g^\sharp = H_1(0) g^\sharp$ be the signed $G_{1,0}$ distance to X_0, then it suffices to show that $\delta_1(w) = H_1^{1/2}\delta_0(w) \in S(1, G_{1,0})$. By chosing

$$\begin{cases} z_1 = H_1^{1/2}(w_1 - \beta(w')) \\ z' = H_1^{1/2} w' \end{cases}$$

as new coordinates, then we find that $G_{1,0}$ transforms to a uniformly bounded C^∞ metric in a neighborhood of the origin. Now $\delta_1(z)$ is $\operatorname{sgn}(z_1)$ times the distance to $z_1 = 0$ with respect to this metric, and this is a C^∞ function in a sufficiently small neighborhood of the origin. Clearly, $|\partial_z \delta_1| \ge c > 0$ in a fixed neighborhood of the origin, so Taylor's formula gives $\delta_1 = \alpha_0 z_1$, where $c/2 \le \alpha_0 \in C^\infty$ in that neighborhood. This completes the proof of the proposition. □

We shall compare our metric with the Beals–Fefferman metric $G = Hg^\sharp$ on $T^*\mathbf{R}^n$, where

$$H^{-1} = 1 + |f| + |f'|^2 \le Ch^{-1}. \tag{3.22}$$

This metric is continuous in t, σ temperate on $T^*\mathbf{R}^n$ and $\sup G/G^\sigma = H^2 \le 1$. We also have that $f \in S(H^{-1}, G)$ (see the proof of Lemma 26.10.2 in [10]).

Proposition 3.9. *We have that $H^{-1} \le CH_1^{-1}$ and $M \le CH_1^{-1}$, which implies that $f \in S(H_1^{-1}, G_1)$.*

Thus, the metric G_1 gives smaller localization errors than the Beals-Fefferman metric.

Proof. First note that by the Cauchy-Schwarz inequality we have

$$M = |f| + |f'|H_1^{-1/2} + \|f''\|H_1^{-1} + h^{1/2}H_1^{-3/2} \le C(H^{-1} + H_1^{-1}).$$

Thus we obtain $M \le CH_1^{-1}$ if we show that $H^{-1} \le CH_1^{-1}$. Observe that we only have to prove this when $|\delta_0| \ll H^{-1/2}$, since else $H^{-1/2} \le C|\delta_0| \le CH_1^{-1/2}$.

If $|\delta_0(w_0)| \le \kappa H^{-1/2}(w_0) \le C\kappa h^{-1/2}$ for $C\kappa < 1$, then there exists $w \in f^{-1}(0)$ such that $|w - w_0| \le \kappa H^{-1/2}(w_0)$. For sufficiently small κ we find from Taylor's formula and the slow variation that $|f(w_0)| \le C\kappa H^{-1}(w_0)$. When $C\kappa \le 1/2$ we obtain that

$$H^{-1}(w_0) \le (1 - C\kappa)^{-1}(1 + |f'(w_0)|^2) \le 2H_1^{-1}(w_0)$$

which completes the proof. $\square$

4. The Weight function

Next, we shall define the weight m_ϱ we shall use, for technical reasons it will depend on a parameter $0 < \varrho \le 1$. The weight will essentially measure how much $t \mapsto \delta_0(t, w)$ changes between the minima of $t \mapsto H_1^{1/2}(t, w)$. Since $H_1^{1/2}$ gives an upper bound on the curvature of the zero set when $H_1^{1/2} \ll 1$, the weight will give a bound on the sign changes of the symbol (see Lemma 4.4). In the following, we let $\langle s \rangle = 1 + |s|$.

Definition 4.1. For $0 < \varrho \le 1$ and $(t_0, w_0) \in \mathbf{R} \times T^*\mathbf{R}^n$ we define

$$m_{\pm,\varrho}(t_0, w_0) = \inf_{\pm(t-t_0)\ge 0} \left\{ \varrho^2|\delta_0(t, w_0) - \delta_0(t_0, w_0)| + H_1^{1/2}(t, w_0)\langle \varrho\delta_0(t, w_0)\rangle \right\} \le 1 \tag{4.1}$$

and

$$m_\varrho = \min(\max(m_{+,\varrho}, m_{-,\varrho}), \varrho^2). \tag{4.2}$$

thus $m_1 = \max(m_{+,1}, m_{-,1})$.

We find that $ch^{1/2} \le m_{\pm,\varrho} \le H_1^{1/2}\langle \varrho\delta_0 \rangle$ so

$$\min(ch^{1/2}, \varrho^2) \le m_\varrho \le \min(H_1^{1/2}\langle \varrho\delta_0 \rangle, \varrho^2). \tag{4.3}$$

by (3.5). Now we have

$$m_1(t_0,w_0) \cong \inf_{t'\le t_0\le t''} \Big\{ \delta_0(t'',w_0) - \delta_0(t',w_0) + H_1^{1/2}(t',w_0)\langle\delta_0(t',w_0)\rangle + H_1^{1/2}(t'',w_0)\langle\delta_0(t'',w_0)\rangle \Big\}$$

and thus $m_1(t_0,w_0) \cong 1$ when $|\delta_0(t,w_0)| \cong H_1^{-1/2}(t,w_0)$ for $t \ge t_0$ or for $t \le t_0$. When $t \mapsto \delta_0(t,w_0)$ is constant, we find that m_ϱ is proportional to the quasi-convex hull of $t \mapsto H_1^{1/2}(t,w_0)$ (i.e., it is convex with respect to the constant functions). The weight also has the "convexity property" given by Proposition 4.7: if $\max_I m_1 \gg \min_I m_1$ on $I = \{\,(t,w):\ a \le t \le b\,\}$, then the variation in t of δ_0 on I is bounded from below: $|\Delta_I \delta_0| \ge c \max_I m_1 > 0$. We shall use the parameter ϱ to obtain suitable norms, but this is just a technicality: all m_ϱ are equivalent according to the following proposition.

Proposition 4.2. *Assume that $\varrho = 1$ or $m_\varrho(t_0,w_0) < \varrho^2 < 1$. For this choice of ϱ there exist $t' \le t_0 \le t''$ such that*

$$|\delta_0(t,w_0) - \delta_0(t_0,w_0)| < \varrho^{-2} m_\varrho(t_0,w_0) \le 1 \tag{4.4}$$

$$H_1^{1/2}(t,w_0)\langle\varrho\delta_0(t,w_0)\rangle < 2m_\varrho(t_0,w_0) \le 2\varrho^2. \tag{4.5}$$

for $t = t'$ and t''. The function $t \mapsto m_\varrho(t,w)$ is regulated such that

$$\varrho_1^2/\varrho_2^2 \le m_{\varrho_1}(t,w)/m_{\varrho_2}(t,w) \le 1 \tag{4.6}$$

when $0 < \varrho_1 \le \varrho_2 \le 1$.

We obtain from the proposition that

$$H_1^{1/2}(t,w_0) < 2m_1(t_0,w_0) \text{ and } |\delta_0(t,w_0)| < 2m_1(t_0,w_0)H_1^{-1/2}(t,w_0)$$

for $t = t'$, t'' corresponding to $m_1(t_0,w_0)$. When $m_1(t_0,w_0) \ll 1$ we may use Proposition 3.8 at (t',w_0) and (t'',w_0). We also obtain from (4.4) that

$$1/2 \le \langle\delta_0(t,w_0)\rangle/\langle\delta_0(t_0,w_0)\rangle \le 2 \tag{4.7}$$

for $t = t'$, t'' corresponding to $m_1(t_0,w_0)$ in Proposition 4.2, which together with (4.3) gives

$$H_1^{-1/2}(t_0,w_0) \le 4\min(H_1^{-1/2}(t',w_0),\ H_1^{-1/2}(t'',w_0)). \tag{4.8}$$

Proof of Proposition 4.2. We have that $m_{\pm,\varrho} \le m_\varrho$ when $m_\varrho < \varrho^2 < 1$ or when $\varrho = 1$. By approximating the limit, we may choose $t'' \ge t_0$ so that

$$\varrho^2(\delta_0(t'',w_0) - \delta_0(t_0,w_0)) + H_1^{1/2}(t'',w_0)\langle\varrho\delta_0(t'',w_0)\rangle < m_{+,\varrho}(t_0,w_0) + ch^{1/2} \tag{4.9}$$

where c is chosen as in (3.5). Then we find $\varrho^2(\delta_0(t'',w_0) - \delta_0(t_0,w_0)) < m_{+,\varrho}(t_0,w_0)$ and $H_1^{1/2}(t'',w_0)\langle\varrho\delta_0(t'',w_0)\rangle < m_{+,\varrho}(t_0,w_0) + ch^{1/2} \le 2m_{+,\varrho}(t_0,w_0)$, since we have $ch^{1/2} \le m_{+,\varrho}(t_0,w_0)$. We similarly obtain these estimates for $m_{-,\varrho}$ with $t' \le t_0$, which gives (4.4)–(4.5).

To prove (4.6) we let $F_\varrho(t,s,w) = \varrho^2|\delta_0(s,w)-\delta_0(t,w)| + H_1^{1/2}(s,w)\langle\varrho\delta_0(s,w)\rangle$. Then we have $F_{\varrho_1} \le F_{\varrho_2}$ and $\varrho_1^2 F_{\varrho_2} \le \varrho_2^2 F_{\varrho_1}$ when $\varrho_1 \le \varrho_2$. Since these estimates are preserved when taking infima and suprema, we obtain (4.6) for $m_{\pm,\varrho_j}$ and m_{ϱ_j}, $j = 1, 2$.

To prove that $t \mapsto m_\varrho(t,w)$ is a regulated function, it suffices to prove that $t \mapsto m_{\pm,\varrho}(t,w)$ is a regulated function since this property is preserved when taking maxima and minima. We note that

$$t \mapsto m_{+,\varrho}(t,w_0) = \inf_{t\le t''} \Big\{ \varrho^2\delta_0(t'',w_0) + H_1^{1/2}(t'',w_0)\langle\varrho\delta_0(t'',w_0)\rangle \Big\} - \varrho^2\delta_0(t,w_0)$$

and since the infimum is non-decreasing and bounded, we find that this gives a regulated function in t. A similar argument works for $m_{-,\varrho}$, which proves the result. ☐

In the following we shall assume the coordinates chosen so that $g^\sharp(w) = |w|^2$. Observe that m_ϱ is not a weight for G_1, but the following proposition shows that it is a weight for $g_\varrho = \varrho^2 g^\sharp$ uniformly in ϱ.

Proposition 4.3. *We find that there exists $C > 0$ such that*

$$m_\varrho(t,w) \le C m_\varrho(t,w_0)(1 + \varrho^2 g^\sharp(w-w_0)) \tag{4.10}$$

uniformly when $0 < \varrho \le 1$, which implies that m_ϱ is a weight for $g_\varrho = \varrho^2 g^\sharp$.

Proof. Since $m_\varrho \le \varrho^2$ we only have to consider the case when

$$m_\varrho(t_0,w_0) < \varrho^2. \tag{4.11}$$

Now, it suffices to show that

$$m_\varrho(t_0,w)/m_\varrho(t_0,w_0) \le C(1+\varrho^2|w-w_0|^2) \qquad \text{when } |w-w_0| \le \varrho m_\varrho^{-1}(t_0,w_0) \tag{4.12}$$

uniformly in $0 < \varrho \le 1$. In fact, when $|w-w_0| > \varrho m_\varrho^{-1}(t_0,w_0)$ we obtain that $\varrho^2|w-w_0|^2 > \varrho^4 m_\varrho^{-2}(t_0,w_0) > m_\varrho(t_0,w)/m_\varrho(t_0,w_0)$ by (4.11). Thus (4.10) is trivially satisfied with $C = 1$ when $|w-w_0| > \varrho m_\varrho^{-1}(t_0,w_0)$, thus in the following we shall assume that $|w-w_0| \le \varrho m_\varrho^{-1}(t_0,w_0)$.

Now, if (4.10) holds for m_{ϱ_0} then it holds for m_ϱ when $\varrho_0 \le \varrho \le 1$, with C replaced by C/ϱ_0^2. Thus, in the following we shall assume $0 < \varrho \le \varrho_0$ is sufficiently small. Let $m_\varrho = m_\varrho(t_0,w_0)$, then for ϱ small enough one can show that

$$|\delta_0(t',w)-\delta_0(t'',w)| \le C_2(\varrho^{-2}m_\varrho + H_0^{1/2}|w'|^2) \qquad \text{when } |w-w_0| \le 2\varrho H_0^{-1/2}, \tag{4.13}$$

where $2\varrho H_0^{-1/2} \ge \varrho m_\varrho^{-1}$ (see the proof of Proposition 6.3 in [7]). We obtain from (4.13) and the monotonicity of $t \mapsto \delta_0(t,w)$ that

$$\varrho^2|\delta_0(t,w)-\delta_0(t_0,w)| \le \varrho^2|\delta_0(t',w)-\delta_0(t'',w)| \le C_2 m_\varrho(1+\varrho^2|w'|^2) \tag{4.14}$$

when $t = t'$, t'' and $|w-w_0| \le 2\varrho H_0^{-1/2}$. Since G_1 is slowly varying we find for small $\varrho > 0$ that $H_1^{1/2}(t,w) \le C_3 H_1^{1/2}(t,w_0)$ when $|w-w_0| \le 2\varrho H_0^{-1/2} \le 2\varrho H_1^{-1/2}(t,w_0)$ and $t = t'$, t''. By the uniform Lipschitz continuity we find

$$\langle\varrho\delta_0(t,w)\rangle \le \langle\varrho\delta_0(t,w_0)\rangle(1+\varrho|w-w_0|) \qquad \text{for } t = t', t'', \tag{4.15}$$

which implies that

$$H_1^{1/2}(t,w)\langle\varrho\delta_0(t,w)\rangle \le C_3 H_1^{1/2}(t,w_0)\langle\varrho\delta_0(t,w_0)\rangle(1+\varrho|w-w_0|) \tag{4.16}$$

when $t = t'$, t'' and $|w - w_0| \le 2\varrho H_0^{-1/2}$. By using (4.4)–(4.5), (4.14), (4.16) and taking the infimum we obtain

$$m_{\pm,\varrho}(t_0,w) \le C_4 m_\varrho(t_0,w_0)(1+\varrho|w-w_0|)^2 \text{ when } |w-w_0| \le \varrho m_\varrho^{-1}(t_0,w_0) \le 2\varrho H_0^{-1/2}$$

uniformly for small ϱ. By taking the maximum and then the minimum, we obtain (4.12) and Proposition 4.3. □

By using the properties of m_ϱ, H_1 and condition $(\overline{\Psi})$ we can prove the following result (see the proof of Proposition 6.4 in [7]).

Lemma 4.4. *There exists $0 < \varrho_0 < 1$ and $c_0 > 0$ such that if $m_1 \le \varrho_0^2$ at $(t_0, w_0) \in \mathbf{R} \times T^*\mathbf{R}^n$, then there exist $g^\sharp$ orthonormal coordinates so that $w_0 = (z_1, 0)$, $|z_1| < |\delta_0(t_0, w_0)| + 1$ and*

$$\operatorname{sgn}(w_1) f(t_0,w) \ge 0 \quad \text{when } |w_1| \ge (1 + H_0^{1/2}|w'|^2)/c_0 \text{ and } |w| \le c_0 H_0^{-1/2} \tag{4.17}$$

where $H_0^{1/2} = \max(H_1^{1/2}(t', w_0), H_1^{1/2}(t'', w_0)) \le 4m_1(t_0, w_0)/\langle\varrho_0\delta_0(t_0, w_0)\rangle \le 4\varrho_0^2$.

In section 5, we shall choose a fixed $\varrho \ll 1$ in order to get invertible operators and suitable norms. In the following, we shall for simplicity only consider m_1, since all the m_ϱ are equivalent when $\varrho \ge c > 0$ by (4.6), this is really no restriction. In order to get lower bounds in terms of the weight m_1 we need the following proposition, which will be important for the proof.

Proposition 4.5. *Let the weight M be given by Definition 3.5. Then there exists $C_0 > 0$ such that*

$$M H_1^{3/2}\langle\delta_0\rangle \le C_0 m_1 \tag{4.18}$$

which gives $S(MH_1^{3/2}, G_1) \subseteq S(m_1\langle\delta_0\rangle^{-1}, g^\sharp)$.

We shall use the following result, for a proof see Proposition 5.2 in [7].

Proposition 4.6. *Let $f \in S(h^{-1}, hg^\sharp)$ and let $H_1^{1/2}$ be given by by Definition 3.4. Assume that*

$$\operatorname{sgn}(w_1) f(w) \ge 0 \quad \text{when } (1 + H_0^{1/2}|w'|^2)/C_0 \le |w_1| \le C_0 H_0^{-1/2} \text{ and } |w'| \le C_0 H_0^{-1/2} \tag{4.19}$$

where $w = (w_1, w')$ and $H_0^{1/2} \ge h^{1/2}/C_0$. If $H_0^{1/2}$ is sufficiently small, then there exist c_1 and C_1 such that

$$|f(0)| \le \partial_{w_1} f(0)\varrho + C_1 h^{1/2}\varrho^3 \tag{4.20}$$

$$\|f''(0)\| \le \partial_{w_1} f(0)/\varrho + C_1 h^{1/2}\varrho \tag{4.21}$$

for any $1 \le \varrho \le c_1 H_0^{-1/2}$.

Proof of Proposition 4.5. We shall put $m_1 = m_1(t_0, w_0)$, note that if $m_1 \ge c > 0$, then $MH_1^{3/2}\langle\delta_0\rangle \le C \le Cm_1/c$ at (t_0, w_0), since $\langle\delta_0\rangle \le H_1^{-1/2}$ and $M \le CH_1^{-1}$ by Proposition 3.9. Thus, we only have to consider the case $m_1 \ll 1$. Let $0 < \varrho_0 < 1$ be given by Lemma 4.4. If $m_1 \le \varrho_0^2$, we may use Lemma 4.4 to obtain $g^\sharp$ orthonormal coordinates so that $|w_0| \le |\delta_0(t_0, w_0)| + 1 \le H_1^{-1/2}(t_0, w_0)$ and f satisfies the conditions in Proposition 4.6 with $H_0^{1/2} = \max(H_1^{1/2}(t', w_0), H_1^{1/2}(t'', w_0)) \le 4m_\varrho(t_0, w_0)/\langle\varrho_0\delta_0(t_0, w_0)\rangle \le 4\varrho_0^2$. In the following, we shall omit the dependence on t_0. Since $\varrho_0\langle\delta_0\rangle \le \langle\varrho_0\delta_0\rangle$ we obtain that

$$H_0^{1/2} < 4\varrho_0^{-1}m_1/\langle\delta_0(w_0)\rangle$$

so we only have to prove the estimate

$$MH_1^{3/2} \le C_1 H_0^{1/2} \qquad \text{at } w = w_0, \tag{4.22}$$

and we shall start by proving this estimate at $w = 0$.

First we observe that if $H_1^{1/2}(0) \le C_0 H_0^{1/2}$ then $M(0)H_1^{3/2}(0) \le CH_1^{1/2}(0) \le CC_0 H_0^{1/2}$ by Proposition 3.9. Thus, in the following we shall assume $H_0^{1/2} \le \varepsilon_0 H_1^{1/2}(0)$ for some $\varepsilon_0 > 0$ to be determined later. >From the definition of M we find

$$MH_1^{3/2} = |f|H_1^{3/2} + |f'|H_1 + \|f''\|H_1^{1/2} + h^{1/2}.$$

When κ_0 is small enough, we find from Proposition 4.6 that $|f(0)| \le C(|f'(0)| + h^{1/2})$ and since $H_1 \le 1$ it suffices to estimate $\|f^{(k)}(0)\|$ for $k = 1, 2$. We obtain from (3.6) that

$$|f'(0)|H_1(0) \le 2\|f''(0)\|H_1^{1/2}(0) + 3h^{1/2}. \tag{4.23}$$

Thus, we only have to estimate $\|f''(0)\|H_1^{1/2}(0)$ in order to obtain (4.22) at $w = 0$. Now by (4.21) we have

$$\|f''(0)\|H_1^{1/2}(0) \le H_1^{1/2}(0)(|f'(0)|/\varrho + C_1 h^{1/2}\varrho)$$

for any $1 \le \varrho \le c_1 H_0^{-1/2}$. Thus, if $\varepsilon_0 \le c_1/4$, we can choose $\varrho = 4H_1^{-1/2}(0) \le 4\varepsilon_0 H_0^{-1/2} \le c_1 H_0^{-1/2}$ which gives

$$\|f''(0)\|H_1^{1/2}(0) \le \frac{1}{4}|f'(0)|H_1(0) + Ch^{1/2} \le \frac{1}{2}\|f''(0)\|H_1^{1/2}(0) + C_2 h^{1/2}. \tag{4.24}$$

by (4.23). This gives $\|f''(0)\|H_1^{1/2}(0) \le 2C_2 h^{1/2} \le 2C_2 C_0 H_0^{1/2}$ and (4.22) at $w = 0$.

It remains to prove the estimate $M(w_0)H_1^{3/2}(w_0) \le CM(0)H_1^{3/2}(0)$ when $|w_0| \le H_1^{-1/2}(w_0)$. By Proposition 3.6 we have that

$$M(w_0) \le CM(0)(1 + H_1^{1/2}(0)|w_0|)^3 \tag{4.25}$$

and

$$H(w_0) \le CH(0)(1 + H_1^{1/2}(w_0)|w_0|)^2. \tag{4.26}$$

In the case $H_1^{1/2}(0) \le H_1^{1/2}(w_0)$ we find that $|w_0| \le H_1^{-1/2}(w_0) \le H_1^{-1/2}(0)$ and thus $M(w_0)H_1^{3/2}(w_0) \le 64C^{5/2}M(0)H_1^{3/2}(0)$ by (4.25)–(4.26). When $H_1^{1/2}(w_0) \le H_1^{1/2}(0)$ we don't use (4.26), instead we find from (4.25) that

$$M(w_0)H_1^{3/2}(w_0) \le CM(0)H_1^{3/2}(0)(H_1^{1/2}(w_0)H_1^{-1/2}(0)+1)^3 \le 8CM(0)H_1^{3/2}(0)$$

since $|w_0| \le H_1^{-1/2}(w_0)$. This completes the proof of the proposition. □

Finally, we shall prove the "convexity property" mentioned earlier.

Proposition 4.7. *Let m_1 be given by Definition 4.1. There exist $\kappa_0 > 1$, $c_0 > 0$ and $\varepsilon_0 > 0$ such that if $\kappa \ge \kappa_0$, $t' < t_0 < t''$ and*

$$m_1(t_0, w_0) = \kappa \max(m_1(t', w_0), m_1(t'', w_0)) \tag{4.27}$$

then we have

$$\delta_0(t'', w) - \delta_0(t', w) \ge c_0 m_1(t_0, w_0) = c_0\kappa \max(m_1(t', w_0), m_1(t'', w_0)) \tag{4.28}$$

when $|w - w_0| \le \varepsilon_0$.

Proof. Since $t_0 < t''$ we have by the triangle inequality

$$\begin{aligned} m_{+,1}(t_0, w_0) &\le \inf_{t'' \le t}(\delta_0(t, w_0) - \delta_0(t_0, w_0) + H_1^{1/2}(t)\langle\delta_0(t, w_0)\rangle) \\ &\le \delta_0(t'', w_0) - \delta_0(t_0, w_0) + m_{+,1}(t'', w_0) \end{aligned}$$

and similarly $m_{-,1}(t_0, w_0) \le \delta_0(t_0, w_0) - \delta_0(t', w_0) + m_{-,1}(t', w_0)$. Since $m_{\pm,1} \le m_1$ we find that

$$\begin{aligned} m_1(t_0, w_0) &= \max(m_{-,1}(t_0, w_0), m_{+,1}(t_0, w_0)) \\ &\le \delta_0(t'', w_0) - \delta_0(t', w_0) + \max(m_1(t', w_0), m_1(t'', w_0)) \end{aligned}$$

which gives (4.28) for $w = w_0$ with $\kappa_0 = 2$ and $c_0 = 1/2$.

If we choose $\varepsilon_0 > 0$ so that $1/C_0 \le m_1(t, w)/m_1(t, w_0) \le C_0$ for $|w - w_0| \le \varepsilon_0$ and all t, then we obtain (4.28) with $\kappa_0 = 2C_0^2$ and $c_0 = (2C_0)^{-1}$. □

5. The Pseudo-Sign

In order to construct a pseudo-sign we shall use the Wick quantization. For $a(x, \xi) \in L^\infty(T^*\mathbf{R}^n)$ we define the Wick quantization:

$$a^{Wick}(x, D_x)u(x) = \int_{T^*\mathbf{R}^n} a(y, \eta)\Sigma^w_{y,\eta}(x, D_x)u(y)\, dyd\eta \qquad u \in \mathcal{S}(\mathbf{R}^n)$$

using the projections $\Sigma^w_{y,\eta}(x, D_x)$ with symbol

$$\Sigma_{y,\eta}(x, \xi) = \pi^{-n}\exp(-g^\sharp(x - y, \xi - \eta)) = \pi^{-n}\exp(-|x - y|^2 - |\xi - \eta|^2).$$

We find that $a^{Wick} : \mathcal{S}(\mathbf{R}^n) \mapsto \mathcal{S}'(\mathbf{R}^n)$ is symmetric on $\mathcal{S}(\mathbf{R}^n)$ if a is real valued,

$$a \ge 0 \quad \text{in } L^\infty(T^*\mathbf{R}^n) \Rightarrow \langle a^{Wick}(x, D_x)u, u\rangle \ge 0 \quad \text{for } u \in \mathcal{S}(\mathbf{R}^n) \tag{5.1}$$

(see [15, Proposition 4.2]). We obtain from the definition that $a^{Wick} = a_0^w$ where

$$a_0(w) = \pi^{-n} \int a(z) \exp(-|w-z|^2)\, dz \tag{5.2}$$

is the Gaussian regularization. Observe that real Wick symbols have real Weyl symbols.

Proposition 5.1. *Assume that $a \in L^\infty(T^*\mathbf{R}^n)$, then $a_0^w = a^{Wick}$ where a_0 is given by (5.2). If $|a| \le CM$ then we find that $a_0 \in S(M, g^\sharp)$. If also $a \in S(M, G_1)$ in a G_1 ball of fixed radius with center w, then $a_0 \cong a$ modulo symbols in $S(H_1 M, G_1)$ in a fixed G_1 neighborhood of w. If $a \ge M$ we obtain $a_0 \ge cM$, and if $a \ge M$ in a G_1 ball of fixed radius with center w then $a_0 \ge cM - CH_1 M$ in a fixed G_1 neighborhood of w, for some constants c, $C > 0$. If $|da| \le C$ almost everywhere, then $a_0 \in S^+(1, g^\sharp)$.*

Proof. Since a is measurable satisfying $|a| \le CM$, we find that $a^{Wick} = a_0^w$ where a_0 is given by (5.2). Since $M(z) \le CM(w)(1 + |z-w|)^3$ by (3.13), we obtain that $a_0(w) = \mathcal{O}(M(w))$. By differentiating on the exponential factor, we find $a_0 \in S(M, g^\sharp)$, and similarly we find that $a_0 \ge M/C$ if $a \ge M$.

If $a \in S(M, H_1\, g^\sharp)$ in a G_1 ball of radius $c > 0$ and center at w, then we write

$$\begin{aligned} a_0(w) = \pi^{-n} &\int_{T^*\mathbf{R}^n} a(z) \exp(-|w-z|^2)\, dz \\ &= \pi^{-n} \int_{|w-z| \le cH_1^{-1/2}(w)/2} a(z) \exp(-|w-z|^2)\, dz \\ &\qquad + \pi^{-n} \int_{|w-z| \ge cH_1^{-1/2}(w)/2} a(z) \exp(-|w-z|^2)\, dz \end{aligned}$$

where the last term is $\mathcal{O}(H_1^N(w)M(w))$ for any N. Thus, after multiplying with a cut-off function, we may assume that $a \in S(M, G_1)$ everywhere. Taylor's formula gives

$$\begin{aligned} a_0(w) = \pi^{-n} &\int_{T^*\mathbf{R}^n} a(w+z) \exp(-|z|^2)\, dz \\ &= a(w) + \pi^{-n} \int_0^1 \int_{T^*\mathbf{R}^n} (1-\theta) \langle a''(w+\theta z)z, z\rangle e^{-|z|^2}\, dz d\theta \end{aligned}$$

where $a'' \in S(MH_1, G_1)$ since $G_1 = H_1 g^\sharp$. Since differentiation commutes with convolution, we find from (3.12)–(3.13) that $a_0(w) \cong a(w)$ modulo symbols in $S(H_1 M, G_1)$. Similarly, we obtain that $a_0 \ge cM$ modulo $S(H_1 M, g^\sharp)$ for some $c > 0$ if $a \ge M$ in a fixed G_1 ball. Since $da_0(w) = \pi^{-n} \int_{T^*\mathbf{R}^n} da(z) \exp(-|w-z|^2)\, dz$, we obtain the last statement. □

Remark 5.2. *If $a(t,w)$ and $g(t,w) \in L^\infty(\mathbf{R}\times T^*\mathbf{R}^n)$ such that $\partial_t a(t,w) \ge g(t,w)$ in $\mathcal{D}'(\mathbf{R})$ for almost all $w \in T^*\mathbf{R}^n$, then we find $\langle \partial_t(a^{Wick})u,u\rangle \ge \langle g^{Wick}u,u\rangle$ in $\mathcal{D}'(\mathbf{R})$ when $u \in \mathcal{S}(\mathbf{R}^n)$.*

In fact, the condition means that

$$-\int a(t,w)\phi'(t)\,dt \ge \int g(t,w)\phi(t)\,dt \qquad 0 \le \phi \in C_0^\infty(\mathbf{R})$$

for almost all $w \in T^*\mathbf{R}^n$, and then (5.1) gives

$$-\int \langle a^{Wick}(t,x,D_x)u,u\rangle\phi'(t)\,dt \ge \int \langle g^{Wick}(t,x,D_x)u,u\rangle\phi(t)\,dt \qquad 0 \le \phi \in C_0^\infty(\mathbf{R})$$

for $u \in \mathcal{S}(\mathbf{R}^n)$.

We are going to use the symbol classes $S(m_\varrho^k, g_\varrho)$ where $g_\varrho = \varrho^2 g^\sharp$ and m_ϱ is given by Definition 4.1. Observe that $S(m_\varrho^k, g_\varrho) = S(m_1^k, g^\sharp)$ for all $0 < \varrho \le 1$. In fact, $g_\varrho = \varrho^2 g^\sharp$ and $m_\varrho \le m_1 \le \varrho^{-2} m_\varrho$ by (4.6). By [2, Corollary 6.7] we can define Sobolev spaces $H(m_\varrho^k, g_\varrho)$ with the following properties: $\mathcal{S}$ is dense in $H(m_\varrho^k, g_\varrho)$, the dual of $H(m_\varrho^k, g_\varrho)$ is naturally identified with $H(m_\varrho^{-k}, g_\varrho)$, and

$$u \in H(m_\varrho^k, g_\varrho) \iff a^w u \in L^2 = H(1, g_\varrho) \qquad \forall\, a \in S(m_\varrho^k, g_\varrho) \tag{5.3}$$

and then $u = a_0^w v$ for some $a_0 \in S(m_\varrho^{-k}, g_\varrho)$ and $v \in L^2$. Observe that $H(m_\varrho^k, g_\varrho) = H(m_1^k, g^\sharp)$ for all $0 < \varrho \le 1$, but not uniformly. We also find from [2, Corollary 4.4] that a^w is bounded as an operator:

$$u \in H(m_\varrho^j, g_\varrho) \mapsto a^w u \in H(m_\varrho^{j-k}, g_\varrho) \qquad \text{when } a \in S(m_\varrho^k, g_\varrho), \tag{5.4}$$

and the bound only depends on the seminorms of a in $S(m_\varrho^k, g_\varrho)$.

Let $\mu_\varrho^w = m_\varrho^{Wick}$, i.e.,

$$\mu_\varrho(t,w) = \pi^{-n}\int_{T^*\mathbf{R}^n} m_\varrho(t,z)\exp(-|w-z|^2)\,dz. \tag{5.5}$$

Since m_ϱ satisfies (4.10) we find from Proposition 5.1 that

$$m_\varrho/c_0 \le \mu_\varrho \in L^\infty(\mathbf{R}, S(m_\varrho, g_\varrho))$$

uniformly for $0 < \varrho \le 1$ for some $c_0 > 0$.

Proposition 5.3. *Assume that the symbol $\mu = \mu_{\varrho_0} \in L^\infty(\mathbf{R}, S(m_1, g^\sharp))$ is given by (5.5) with $\varrho = \varrho_0 \ll 1$, thus $\mu^w = m_{\varrho_0}^{Wick} \le m_1^{Wick}$. Then there exist positive constants c_0, c_1 and C_0 such that*

$$c_0 h^{1/2}\|u\|^2 \le c_1\|u\|^2_{H(m_1^{1/2})} \le \langle \mu^w u,u\rangle \le C_0\|u\|^2_{H(m_1^{1/2})}. \tag{5.6}$$

The proof relies on the fact that when $0 < \varrho \ll 1$ we have

$$(m_\varrho^{1/2})^w (m_\varrho^{-1/2})^w = \gamma_\varrho^w \quad \text{is uniformly invertible in } L^2 \tag{5.7}$$

$$\frac{1}{2} \le (m_\varrho^{-1/2})^w m_\varrho^w (m_\varrho^{-1/2})^w \le 2 \qquad \text{in } L^2 \tag{5.8}$$

when $|t| \le 1$ (see the proof of [7, Proposition 7.4]).

Next, we shall construct a perturbation $B(t,w) = \delta_0(t,w) + \varrho_0(t,w)$ of δ_0 so that $b^w = B^{Wick}$ could be used as the pseudo-sign in Proposition 2.9.

Proposition 5.4. *Assume that δ_0 is given by Definition 3.2 and m_1 is given by Definition 4.1. Then there exist a positive constant C_1 and a real valued $\varrho_0(t,w) \in L^\infty(\mathbf{R}\times T^*\mathbf{R}^n)$ such that*

$$|\varrho_0| \le C_1 m_1 \tag{5.9}$$
$$\partial_t(\delta_0 + \varrho_0) \ge m_1/C_1 \tag{5.10}$$

in $\mathcal{D}'(\mathbf{R})$ when $|t| < 1$. We also have that $t \mapsto \varrho_0(t,w)$ is a regulated function, $\forall\, w \in T^*\mathbf{R}^n$, and $w \mapsto \varrho_0(t,w) \in \mathrm{Lip}(T^*\mathbf{R}^n)$ uniformly for almost all $|t| \le 1$.

The proof is long and technical (see the proof of [7, Proposition 8.1]), but the idea is as follows. When $t \mapsto m_1(t,w)$ has a approximate minimum at $t = t_0$ in the sense that $m(s) \le Cm(t)$ when $t \le s \le t_0$ or $t_0 \le s \le t$, we may take $\varrho_0(t,w) = c\int_{t_0}^t m_1(s,w)\,ds$ since $t \mapsto \delta_0(t,w)$ is non-decreasing. In general, we have to split the interval $[-1,1]$ into subintervals where $t \mapsto m_1(t,w)$ has approximate maximum and minimum, and use the “convexity property” of $t \mapsto \delta_0(t,w)$ given by Proposition 4.7 in order to interpolate δ_0 at the approximate maxima of $t \mapsto \delta_0(t,w)$.

By Proposition 5.4 and Remark 5.2 we obtain lower bounds on $\partial_t B^{Wick}$ if $B = \delta_0 + \varrho_0$. But in order to prove Proposition 2.9 we also have to obtain lower bounds on $\operatorname{Re} B^{Wick} f^w$. To obtain that, we have to compute the Weyl symbol for the pseudo-sign B^{Wick}.

Proposition 5.5. *Let $B = \delta_0 + \varrho_0$, where δ_0 is given by Definition 3.2 and $\varrho_0(t,w)$ is the real valued symbol given by Proposition 5.4, satisfying $|\varrho_0(t,w)| \le Cm_1(t,w)$ for almost all $|t| \le 1$. Then we find*

$$B^{Wick} = b^w \qquad |t| \le 1$$

where $b = \delta_1 + \varrho_1 \in S(H_1^{-1/2}, g^\sharp)\bigcap S^+(1,g^\sharp)$ is real valued and regulated in t, and $\varrho_1 \in S(m_1, g^\sharp) \subseteq S(H_1^{1/2}\langle\delta_0\rangle, g^\sharp)$ for almost all $|t| \le 1$. There also exists a positive constant κ_2 with the following properties. For any $\lambda > 0$, there exists $c_\lambda > 0$ such that if $|\delta_0| \ge \lambda H_1^{-1/2}$ and $H_1^{1/2} \le c_\lambda$ then $|b| \ge \kappa_2 \lambda H_1^{-1/2}$. If $H_1^{1/2}(t,w_0) \le \kappa_2$ and $|\delta_0(t,w_0)| \le \kappa_2 H_1^{-1/2}(t,w_0)$ then we have $S(H_1^{-1/2}, G_1) \ni \delta_1(t,w) = \delta_0(t,w) + \varrho_2(t,w)$ when $|w - w_0| \le \kappa_2 H_1^{-1/2}(t,w_0)$ with real valued $\varrho_2(t,w) \in S(H_1^{1/2}, G_1)$.

Proof. Let $\delta_0^{Wick} = \delta_1^w$ and $\varrho_0^{Wick} = \varrho_1^w$. Since $|\delta_0| \le CH_1^{-1/2}$, $|\varrho_0| \le Cm_1$ and the symbols are real valued, we obtain from Proposition 5.1 and (4.3) that $\delta_1 \in S(H_1^{-1/2}, g^\sharp)$ and $\varrho_1 \in S(m_1, g^\sharp) \subseteq S(H_1^{1/2}\langle\delta_0\rangle, g^\sharp)$ are real valued for almost all $|t| \le 1$. Observe that $m_1 \le 1$, and since $|\delta_0'| \le 1$ almost everywhere we find that $b \in S^+(1,g^\sharp)$ for almost all $|t| \le 1$ by Proposition 5.1. Since $\delta_0(t,w)$ and $\varrho_0(t,w)$ are regulated in t, we find from (5.2) that the same holds for $\delta_1(t,w)$ and $\varrho_1(t,w)$.

When $|\delta_0| \ge \lambda H_1^{-1/2}$ at (t,w), $\lambda > 0$, then by the Lipschitz continuity and slow variation we find that $|\delta_0| \ge \lambda H_1^{-1/2}/C_0$ in a G_1 neighborhood ω of (t,w) (depending on λ). Since $|\varrho_0| \le CH_1^{1/2}\langle\delta_0\rangle$ we find by the slow variation that $|\delta_0 + \varrho_0| \ge \lambda H_1^{-1/2}/2C_0$ in ω when $H_1^{1/2}(t,w)$ is small enough. Proposition 5.1 gives $|b| \ge c\lambda H_1^{-1/2}/2C_0 - C\lambda H_1^{1/2}/2C_0 \ge c\lambda H_1^{-1/2}/3C_0$ at (t,w) when $H_1^{1/2}(t,w)$ is small enough.

If $|\delta_0| \le \kappa_2 H_1^{-1/2}$ and $H_1^{1/2} \le \kappa_2$ for sufficiently small $\kappa_2 > 0$, then $|\delta_0| \le C_0\kappa_2 H_1^{-1/2}$ and $H_1^{1/2} \le C_0\kappa_2$ in a fixed G_1 neighborhood. Thus, for $\kappa_2 \ll 1$ we obtain that $\delta_0 \in S(H_1^{-1/2}, G_1)$ in a fixed G_1 neighborhood. Then we obtain the last statement from Proposition 5.1, which completes the proof. $\square$

Next, we shall obtain lower bounds on $\operatorname{Re} B^{Wick} f^w = \operatorname{Re} b^w f^w$, and finally prove Proposition 2.9.

Proposition 5.6. *Assume that $b = \delta_1 + \varrho_1$ is given by Proposition 5.5. Then we have*

$$\operatorname{Re}\langle (b^w f^w)\big|_t u, u\rangle \ge \langle C_t^w u, u\rangle \qquad \forall\, u \in C_0^\infty(\mathbf{R}^n) \qquad \textit{for almost all } |t| \le 1 \tag{5.11}$$

where $C_t \in S(m_1(t), g^\sharp)$ has uniformly bounded seminorms.

The proposition is proved by localizing with respect to the metric G_1 for fixed t. Observe that we may ignore terms in $\operatorname{Op} S(MH_1^{3/2}\langle\delta_0\rangle, g^\sharp) \subseteq \operatorname{Op} S(m_1, g^\sharp)$ by Proposition 4.5, which makes the localization possible. Also, when $H_1 \cong 1$ we have $bf \in S(MH_1^N, g^\sharp)$ for any N, thus we may assume $H_1 \ll 1$.

By the slow variation of G_1 and the uniform Lipschitz continuity of $w \mapsto \delta_0(w)$, we may consider the domains where $|\delta_0| \gtrless \kappa H_1^{-1/2}$ for $\kappa \ll 1$. When $|\delta_0| \ge \kappa H_1^{-1/2}$ and $H_1 \ll 1$ we use that bf is a product of a non-negative symbol and an elliptic symbol. Moreover, we find that $MH_1^{3/2}\langle\delta_0\rangle \cong MH_1$ in this case, so by perturbing the Fefferman-Phong estimate for f^w we obtain the lower bounds in this case.

When $|\delta_0| \ll H_1^{-1/2}$ and $H_1 \ll 1$ we find that $\delta_0 \in S(H_1^{-1/2}, G_1)$ but $bf \not\ge 0$. By completing the square and taking an approximate square root, we obtain the lower bounds by using the calculus (see the proof of Theorem 9.1 in [7]).

Proof of Proposition 2.9. Let $B^{Wick} = (\delta_0 + \varrho_0)^{Wick}$ be the pseudo-sign, where $\delta_0 + \varrho_0$ is given by Proposition 5.4. We find that $B^{Wick} = b^w = (\delta_1 + \varrho_1)^w$ where $b(t,w) \in L^\infty(\mathbf{R}, S(H_1^{-1/2}, g^\sharp) + S^+(1, g^\sharp))$ is given by Proposition 5.5 for $|t| \le 1$. Now $\partial_t(\delta_0 + \varrho_0) \ge m_1/C_1$ in $\mathcal{D}'(\mathbf{R})$ when $|t| < 1$ by Proposition 5.4. Let $\mu^w \in L^\infty(\mathbf{R}, \operatorname{Op} S(m_1, g^\sharp))$ be given by Proposition 5.3, then $m_1^{Wick} \ge \mu^w$. Thus we find by Remark 5.2 that

$$\partial_t\langle b^w u, u\rangle = \langle \partial_t B^{Wick} u, u\rangle \ge C_1^{-1}\langle \mu^w u, u\rangle \qquad \text{in } \mathcal{D}'(\mathbf{R}) \tag{5.12}$$

when $u \in C_0^\infty(\mathbf{R}^n)$. We obtain from Proposition 5.3 that there exist positive constants c_0 and c_1 so that

$$\langle \mu^w u, u\rangle \ge c_1\|u\|^2_{H(m_1^{1/2})} \ge c_0 h^{1/2}\|u\|^2 \qquad u \in C_0^\infty(\mathbf{R}^n). \tag{5.13}$$

Here $\|u\|_{H(m_1^{1/2})}$ is the norm of the Sobolev space $H(m_1^{1/2}, g^\sharp) = H(m_1^{1/2})$ given by (5.3) with $\varrho = 1$ and $k = 1/2$. By Proposition 5.6 we find for almost all $|t| \le 1$ that

$$\operatorname{Re}\langle (B^{Wick} f^w)\big|_t u, u\rangle = \operatorname{Re}\langle (b^w f^w)\big|_t u, u\rangle \ge \langle C_t^w u, u\rangle \quad u \in C_0^\infty(\mathbf{R}^n) \tag{5.14}$$

with $C_t \in S(m_1(t), g^\sharp)$ uniformly. Thus we obtain from (5.4) and duality that there exists a positive constant c_2 such that

$$|\langle C_t^w u, u\rangle| \le \|u\|_{H(m_1^{1/2})} \|C_t^w u\|_{H(m_1^{-1/2})} \le c_2 \|u\|^2_{H(m_1^{1/2})} \le c_2 \langle \mu^w u, u\rangle / c_1 \qquad (5.15)$$

for $u \in C_0^\infty(\mathbf{R}^n)$ and almost all $|t| \le 1$. We obtain Proposition 2.9 from (5.12)–(5.15), which completes the proof of Theorem 1.1 and the Nirenberg-Treves conjecture. □

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