Journées *Équations aux dérivées partielles*
Plestin-les-grèves, 5–8 juin 2002
GDR 1151 (CNRS)

On exponential convergence to a stationary measure for a class of random dynamical systems

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Abstract

For a class of random dynamical systems which describe dissipative nonlinear PDEs perturbed by a bounded random kick-force, I propose a "direct proof" of the uniqueness of the stationary measure and exponential convergence of solutions to this measure, by showing that the transfer-operator, acting in the space of probability measures given the Kantorovich metric, defines a contraction of this space.

0. Introduction

In the papers [3, 4, 5] my collaborators and I considered a special class of random dynamical systems (RDSs) which describes dissipative nonlinear PDEs (e.g., the 2D Navier-Stokes equations), perturbed by a bounded random kick-force. In [3] we proved that these systems have unique stationary measure, by reducing this problem to the problem of uniqueness of a Gibbs measure for a class of 1D Gibbs systems. In [4, 5] we developed a coupling approach to study the systems under discussion. This approach gives a shorter proof of the uniqueness and implies that any solution of the system exponentially fast converges in distribution to the stationary measure.

The goal of this work is to present a "direct proof" of the uniqueness and of the exponential convergence by showing that the transfer-operator, corresponding to an RDS as above and acting in the space of probability measures, given the Kantorovich(–Wasserstein) metric, defines a contraction of this space.

The proof presented in this work can be treated as re-interpreting of the arguments from [4, 5]: it is based on the coupling-approach and uses essentially Lemma 3.2 from [4] (which is the heart of the proof of [4]). In addition to the coupling techniques, we now use some ideas, originated in the works of Kantorovich on the mass-transfer problem in 1940's, see [2, 1].

Due to short size of this paper, we practically do not discuss applications of the results obtained, as well as their relation to works of other mathematicians. For

MSC 2000 : 37H99, 35Q30.
Keywords : random dynamical systems, Kantorovich metric, Navier-Stokes equations.

all this information readers are referred to rather detailed introductions to [3, 4] (post-script files of these works, as well as of [5], can be obtained from the author's web-page www.ma.hw.ac.uk/~kuksin).

We keep notations of [4, 5] and for convenience repeat them now:

Notations. We denote by $(\Omega, \mathcal{F}, \mathbb{P})$ and $(\Omega', \mathcal{F}', \mathbb{P}')$ different probability spaces, and abbreviate them to Ω and Ω', respectively. All metric spaces are given Borel sigma-algebras. $\mathcal{D}(\cdot)$ signifies the distribution of a random variable.

A Hilbert space H with a norm $\|\cdot\|$ is fixed in this work. We use the following notations for objects, related to H:

$\mathcal{B} = \mathcal{B}(H)$ – sigma-algebra of Borel subsets of H;

C_b – the space of bounded continuous functions on H, given the sup-norm;

$\mathcal{P}$ – the space of probability Borel measures on H;

$\mathcal{P}(A)$ – measures from $\mathcal{P}$, supported by a subset $A \subset H$;

$B(R)$ – the closed ball of radius R in H, centred at the origin.

Acknowledgements. This paper is based on my talk at the conference in Plestin–les–Grèves (June 2001); I am thankful to the organisers for inviting me there. It was written during my visit to IHES in June, 2001 and was typed there; I sincerely thank the institute for hospitality.

1. A class of random dynamical systems

Let H be a Hilbert space with a norm $\|\cdot\|$ and an orthonormal basis $\{e_j\}$, and let $S : H \to H$ be a continuous map such that $S(0) = 0$ and S satisfies some conditions, specified below.

Let $\{\eta_k, k \in \mathbb{Z}\}$ be a sequence of i.i.d. random variables $\Omega \to H$ of the form

$$\eta_k = \eta_k^\omega = \sum_{j=1}^{\infty} b_j \xi_{jk} e_j, \tag{1.1}$$

where $b_j \geq 0$ are constants and $\sum b_j^2 < \infty$. It is assumed that $\{\xi_{jk} = \xi_{jk}^\omega\}$ are independent random variables such that $|\xi_{jk}| \leq 1$ for all j, k, ω, and

$$\mathcal{D}(\xi_{jk}) = \ p_j(r)\, dr \qquad \forall j, k.$$

Here $p_1, p_2, \dots$ are functions of bounded variation, supported by the segment $[-1, 1]$, and

$$\int_{-\varepsilon}^{\varepsilon} p_j(r)\, dr > 0 \qquad \forall j \geq 1, \quad \varepsilon > 0. \tag{1.2}$$

We consider the following *random dynamical system* (RDS) in H:

$$u(k) = S(u(k-1)) + \eta_k =: F_k^\omega(u(k-1)) \qquad k \geq 1. \tag{1.3}$$

This RDS defines a family of Markov chains in H with the transition function

$$P(k, v, \Gamma) = \mathbb{P}\{u(k) \in \Gamma\}, \qquad \Gamma \in \mathcal{B}(H),$$

where $u(\cdot) = u(\cdot; v)$ is a solution for (1.3) such that $u(0) = v$. Let $\{\mathfrak{S}_k\}$ and $\{\mathfrak{S}_k^*\}$ be the corresponding Markov semigroups, acting in the space C_b of bounded continuous functions on H, and in the space $\mathcal{P}$ of probability Borel measures, respectively:

$$\mathfrak{S}_k f(v) = \mathbb{E} f(u(k; v)), \qquad f \in C_b,$$
$$\mathfrak{S}_k^* \mu(\Gamma) = \int_H \mathbb{P}\{u(k; v) \in \Gamma\} \, \mu(dv), \qquad \mu \in \mathcal{P},$$

where u is the solution for (1.3) as above.

For any $v \in H$ and $k = 0, 1, \dots$ we abbreviate

$$\mu_v(k) = P(k, v, \cdot) = \mathcal{D}(u(k; v)).$$

Now we impose some assumptions on the map S. The"right" ones are given in [4], see there conditions A-C. Below we replace them by shorter and stronger conditions A′) and B′). The new conditions hold for the RDS which corresponds to the 2D Navier-Stokes equations (see the example below). The proof of the Main Theorem which we present below works under the conditions A-C but becomes somewhat longer, and the notations become more cumbersome.

A′) The map S is Lipschitz uniformly on bounded subsets of H, and there exists a positive constant $\gamma_0 < 1$ such that

$$\|S(u)\| \le \gamma_0 \|u\| \quad \forall u \in H. \tag{1.4}$$

B′) For any $R > 0$ there is a sequence $\gamma_N(R) > 0$ $(N \ge 1)$ which converges to zero as $N \to \infty$, such that

$$\Big\| Q_N\big(S(u_1) - S(u_2)\big) \Big\| \le \gamma_N(R) \, \|u_1 - u_2\| \quad \text{for all} \quad u_1, u_2 \in B(R).$$

Here Q_N stands for the orthogonal projector $H \to \overline{\text{span}}\{e_N, e_{N+1}, \dots\}$.

Example. Let us consider the 2D Navier-Stokes equations, perturbed by a random kick-force η:

$$\dot{u} - \nu \Delta u + (u \cdot \nabla) u + \nabla p = \eta(t, x) \equiv \sum_{k \in \mathbb{Z}} \eta_k(x) \delta(t - k), \tag{1.5}$$
$$\text{div} u = 0, \quad \int u \, dx \equiv \int \eta \, dx \equiv 0; \qquad x \in \mathbb{T}^2.$$

Let H be the L^2-space of divergence-free vector fields on $\mathbb{T}^2$ with zero space-average, and let $\{e_j\}$ be the usual trigonometric basis of H. Let us assume that the kicks η_k are random variables in H having the form (1.1) and satisfying (1.2). Normalising solutions $u(t) \in H$ of (1.5) to be continuous from the right, we observe that the equation can be written in the form (1.3), where $u(k) = u(k, \cdot) \in H$, $k \in \mathbb{Z}$, and the operator S is the time-one shift along trajectories of the free Navier-Stokes system. The condition A′) obviously holds with $\gamma_0 = e^{-\lambda}$, where λ is the minimal eigenvalue of $-\nu\Delta$ in H. It is also well known that S satisfies B′), see e.g. [3].

A measure $\mu \in \mathcal{P}$ is called a *stationary measure* for the RDS (1.3) if $\mathfrak{S}_k^* \mu = \mu$ for all k. The goal of this work is to prove the following result:

Theorem 1. *There exists a constant $N \geq 1$ such that if*

$$b_j \neq 0 \qquad \forall j \leq N, \tag{1.6}$$

then the RDS (1.3) has a unique stationary measure μ. Moreover, there exists a constant $\kappa \in (0,1)$ such that

$$|(\mu_u(t), f) - (\mu, f)| \leq C\kappa^t \quad \textit{for} \quad t = 1, 2, \dots, \tag{1.7}$$

for every Lipschitz function f on H such that $|f| \leq 1$ and $\operatorname{Lip} f \leq 1$. The constant C depends only on $\|u\|$.

2. Preliminaries

2.1. Estimates for solutions.

Since $|\xi_{jk}| \leq 1$, then

$$\|\eta_k^\omega\| \leq K_1 = (b_1^2 + b_2^2 + \dots)^{1/2} < \infty \quad \text{for all } k \text{ and } \omega. \tag{2.1}$$

So

$$\|F_k^\omega(u)\| \leq \gamma_0 \|u\| + K_1,$$

and any ball $B(R)$ with $R \geq K_1/(1-\gamma_0)$ is invariant for the RDS (1.3) (a set $A \subset \mathcal{B}(H)$ is said to be *invariant* for (1.3) if $P(k, u, A) = 1$ for $k \geq 0$ and $u \in A$). The same estimate above implies that

$$\|u(k; v)\| \leq \gamma_0^k \|v\| + K_1(1 + \dots + \gamma_0^{k-1}) \leq \gamma_0^k \|v\| + \frac{K_1}{1-\gamma_0}, \tag{2.2}$$

for all $k \geq 0$, $v \in H$ and all ω.

2.2. The coupling.

Let $\mu_1, \mu_2 \in \mathcal{P}$.

Definition. A pair of random variables ξ_1, ξ_2, defined on the same probability space and valued in H, is called a *coupling* for (μ_1, μ_2) if $\mathcal{D}\xi_1 = \mu_1$ and $\mathcal{D}\xi_2 = \mu_2$.

For basic results on the coupling see [6] and Appendix in [4].

The following lemma, proved in [4], Lemma 3.2, claims that measures $\mu_{u_1}(1)$, $\mu_{u_2}(1)$ admit a coupling which possesses some special properties if $\|u_1 - u_2\| \ll 1$.

Let us take any $R \geq 1$.

Lemma 1. *There is a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, an integer $N = N(R) \geq 1$ and a constant $C_* = C_*(R) > 0$ such that if (1.6) holds, then for any $u_1, u_2 \in B(R)$ the measures $\mu_{u_1}(1), \mu_{u_2}(1)$ admit a coupling (V_1, V_2), $V_j = V_j(u_1, u_2; \omega)$, with the following properties:*

(i) the maps $V_1, V_2 : B(R)^2 \times \Omega \to H$ are measurable;

(ii) denoting $d = \|u_1 - u_2\|$, we have

$$\mathbb{P}\{\|V_1 - V_2\| \geq d/2\} \leq C_* d. \tag{2.3}$$

2.3. A metric on $\mathcal{P}$.

Let us take any number

$$R' > K_1/(1-\gamma_0).$$

We fix it from now on and abbreviate $B(R') = B$. Due to the results of section 2.1, the set B is invariant for the RDS (1.3). Next we take any $\gamma_1 \in (\gamma_0, 1)$ and any positive d_0 such that

$$d_0 \le \min\Big\{\frac{1}{4C_*}, \frac{1-\gamma_1}{2C_*}, 1\Big\}, \tag{2.4}$$

where $C_* = C_*(R')$ (see Lemma 1). For $k \in \mathbb{Z}$ we set $d_k = \gamma_1^k d_0$. We may assume that d_0 and R' are chosen such that $d_{-L} = R'$ for some $L \ge 1$. Below we consider the numbers d_k with $k \ge -L$ only.

Let us introduce in the space H equivalent metric d:

$$d(u_1, u_2) = \|u_1 - u_2\| \wedge d_0,$$

and consider the set $\mathcal{O} \subset C_b$, formed by all functions f such that

$$|f(u_1) - f(u_2)| \le d(u_1, u_2) \qquad \text{for all } u_1,\ u_2.$$

Clearly,

$$\frac{1}{2} d_0 f \in \mathcal{O} \qquad \text{if } |f| \le 1 \text{ and } \operatorname{Lip} f \le 1. \tag{2.5}$$

For any two measures $\mu_1, \mu_2 \in \mathcal{P}$ we define the *Kantorovich distance* $d_K(\mu_1, \mu_2)$ as

$$d_K(\mu_1, \mu_2) = \sup_{g \in \mathcal{O}} \{(\mu_1 - \mu_2, g)\}. \tag{2.6}$$

It is known that the space $\mathcal{P}$ is complete with respect to this distance (see [2], [1]), and it is easy to see that $\mathcal{P}(B)$ is a closed subset of $\mathcal{P}$.

We remind that the set $B = B(R')$ is invariant for (1.3).

Lemma 2. *Suppose that there exists a sequence $\zeta_k \to 0$ such that for $k \ge 1$ and $u, v \in B$ we have $d_K(\mu_u(k), \mu_v(k)) \le \zeta_k$. Then there exists a unique measure $\mu \in \mathcal{P}(B)$, such that*

$$d_K(\mu_u(k), \mu) \le \zeta_k \qquad \textit{for } k \ge 1 \textit{ and } u \in B. \tag{2.7}$$

Proof. Let us take any function $f \in \mathcal{O}$. Using the Chapman-Kolmogorov relation and the assumption of the lemma, for $\ell \ge k \ge 0$ and $u, v \in B$ we have:

$$(\mu_v(\ell) - \mu_u(k), f) = \int_B P(\ell - k, v, dz) \int_B (P(k, z, dw) - P(k, u, dw)) f(w)$$
$$\le \zeta_k \int_B P(\ell - k, v, dz) = \zeta_k. \tag{2.8}$$

Hence, $d_K(\mu_v(\ell), \mu_u(k)) \le \zeta_k$. Since the space $(\mathcal{P}, d_K)$ is complete, then there exists a unique measure $\mu \in \mathcal{P}$ such that $d_K(\mu_u(k), \mu) \to 0$ as $k \to \infty$, for every $u \in B$. Passing to limit in (2.8) as $\ell \to \infty$ we recover (2.7). It is clear that $\operatorname{supp} \mu \subset B$. So $\mu \in \mathcal{P}(B)$ and the lemma is proved. □

3. A Kantorovich-type functional

First we shall construct a special bounded measurable function f_K on $B \times B$, vanishing on the diagonal. To define the function, we consider partition of $B \times B$ to sets Q_ℓ, $-L \le \ell \le \infty$. Here Q_∞ is the diagonal of $B \times B$,

$$Q_r = \left\{(u_1, u_2) \in B \times B \;\middle|\; d_{r+1} < \|u_1 - u_2\| \le d_r\right\}$$

if $0 \le r < \infty$, and

$$Q_r = \left\{(u_1, u_2) \in B \times B \;\middle|\; \|u_1 - u_2\| > d_0, \quad \frac{1}{2}\gamma_1 d_r < \|u_1\| \vee \|u_2\| \le \frac{1}{2} d_r\right\}$$

if $-L \le r < 0$.

Now we define the function f_K:

$$f_K(u_1, u_2) = \begin{cases} d_r, & \text{if } (u_1, u_2) \in Q_r,\ 0 \le r \le \infty \\ \widetilde{d}_\ell, & \text{if } (u_1, u_2) \in Q_\ell,\ \ell < 0 \end{cases}$$

where $d_\infty = 0$ and the numbers $\{\widetilde{d}_\ell\}$ such that

$$d_0 \le \widetilde{d}_{-1} \le \ldots \le \widetilde{d}_{-L} \tag{3.1}$$

are constructed below. Clearly,

$$d_{-L} \ge f_K(u_1, u_2) \ge d(u_1, u_2) \tag{3.2}$$

for all u_1, u_2.

For any pair of measures $\mu_1, \mu_2 \in \mathcal{P}(B)$ we define a Kantorovich-type functional $\mathcal{K}(\mu_1, \mu_2)$ as follows:

$$\mathcal{K}(\mu_1, \mu_2) = \inf\{\mathbb{E} f_K(U_1, U_2)\}, \tag{3.3}$$

where the infimum is taken over all couplings (U_1, U_2) for (μ_1, μ_2).

Everywhere below (and in Theorem 1) $N = N(R')$ is the constant from Lemma 1.

Theorem 2. *Let us assume that the assumption (1.6) holds. Then there exists $\kappa < 1$ such that*

$$\mathcal{K}\big(\mathfrak{S}_1^*(\mu_1), \mathfrak{S}_1^*(\mu_2)\big) \le \kappa \mathcal{K}(\mu_1, \mu_2) \tag{3.4}$$

for all $\mu_1, \mu_2 \in \mathcal{P}(B)$ (provided that the numbers $\widetilde{d}_{-1}, \ldots, \widetilde{d}_{-L}$ are chosen accordingly).

The theorem is proved in the next section. Now we continue to study the RDS (1.3), taking the theorem for granted.

Let (U_1, U_2) be a coupling for (μ_1, μ_2). Using (3.2), for any $g \in \mathcal{O}$ we get:

$$(\mu_1 - \mu_2, g) = \mathbb{E}(g(U_1) - g(U_2)) \le \mathbb{E} d(U_1, U_2) \le \mathbb{E} f_K(U_1, U_2).$$

Comparing this estimate with the definitions (2.6) and (3.3) we find that[1]

$$d_K(\mu_1, \mu_2) \le \mathcal{K}(\mu_1, \mu_2). \tag{3.5}$$

[1] A celebrated theorem of Kantorovich says that the inequality (3.5) transforms to the equality if in (3.3) we replace $f(U_1, U_2)$ by $d(U_1, U_2)$. See in [1, 2].

Let us take any $u_1, u_2 \in B$. Then $\mu_{u_1}(k), \mu_{u_2}(k) \in \mathcal{P}(B)$ for all $k \geq 0$. Iterating (3.4) and using (3.5) together with the first inequality in (3.2), we obtain

$$\begin{aligned} d_K(\mu_{u_1}(k), \mu_{u_2}(k)) &\leq \mathcal{K}(\mu_{u_1}(k), \mu_{u_2}(k)) \\ &\leq \kappa^k \mathcal{K}(\mu_{u_1}(0), \mu_{u_2}(0)) \\ &= \kappa^k f_K(u_1, u_2) \leq \kappa^k \widetilde{d}_{-L}. \end{aligned}$$

Applying Lemma 2 we get that there exists a unique measure $\mu \in \mathcal{P}(B)$ such that $d_K(\mu_u(k), \mu) \leq \kappa^k \widetilde{d}_{-L}$ for all $k \geq 0$, $u \in B$.

Let us take a measure $\nu \in \mathcal{P}(B)$. For a function $f \in \mathcal{O}$ we have:

$$(\mathfrak{S}_k^*(\nu) - \mu, f) = \int (\mu_u(k) - \mu, f)\, d\nu(u) \leq \kappa^k \widetilde{d}_{-L}.$$

Hence,

$$d_K(\mathfrak{S}_k^*(\nu), \mu) \leq \kappa^k \widetilde{d}_{-L} \qquad \forall k \geq 0, \quad \nu \in \mathcal{P}(B). \tag{3.6}$$

Now let us take any $u \in H$. Due to (2.2) there exists $\ell = \ell(\|u\|)$ such that $\mu_u(\ell) \in \mathcal{P}(B)$. Since $\mu_u(k+\ell) = \mathfrak{S}_k^* \mu_u(\ell)$, then denoting $k + \ell = t$ we get from (3.6) that

$$d_K(\mu_u(t), \mu) \leq \kappa^{t-\ell} \widetilde{d}_{-L}, \tag{3.7}$$

for any $u \in H$, where $\ell = \ell(\|u\|)$. Due to (2.5) and (2.6) with $g = \pm \frac{d_0}{2} f$, (3.7) implies (1.7) with $C = \widetilde{d}_{-L} \kappa^{-\ell}$.

The estimate (1.7) easily implies that μ is the unique stationary measure. Indeed, if $\widetilde{\mu}$ is another one, then for any function f as in (1.7) we have

$$\begin{aligned} |(\widetilde{\mu}, f) - (\mu, f)| &= \left| \int (\mu_u(k), f)\, \widetilde{\mu}(du) - \int (\mu, f)\, \widetilde{\mu}(du) \right| \\ &\leq \int |(\mu_u(k) - \mu, f)|\, \widetilde{\mu}(du). \end{aligned} \tag{3.8}$$

The integrand is bounded by two and goes to zero as $k \to \infty$ due to (1.7). So the integral goes to zero as $k \to \infty$ as well and $(\widetilde{\mu}, f) = (\mu, f)$ for all functions as above. Hence, $\mu = \widetilde{\mu}$.

Theorem 1 is proved.

4. Proof of Theorem 2

Let us take any $A' > \mathcal{K}(\mu_1, \mu_2)$. Then there exists a coupling (U_1', U_2') for (μ_1, μ_2) such that $\mathbb{E} f_K(U_1', U_2') \leq A'$. The random variables U_1', U_2' are defined on some probability space $(\Omega', \mathcal{F}', \mathbb{P}')$. Since supports of μ_1, μ_2 belong to B, we may assume that $U_1', U_2' \in B$ for all ω'. Now applying Lemma 1 with $R = R'$, we find measurable maps $V_1, V_2 : B^2 \times \Omega \to H$ such that

$$\mathcal{D}(V_j(u_1, u_2; \cdot)) = \mu_{u_j}(1) = P(1, u_j, \cdot) \tag{4.1}$$

for $j=1,2$. Let us consider the following random variables U_1, U_2, defined on the probability space $\Omega\times\Omega'$:

$$U_j(\omega,\omega')=V_j(U_1'(\omega'),U_2'(\omega');\omega),\qquad j=1,2.$$

Let us take any $f\in C_b$. Using (4.1) and the fact that $\mathcal{D}(U_1')=\mu_1$, we get:

$$\begin{aligned}\mathbb{E}^{\omega,\omega'}f(U_1)&=\mathbb{E}^{\omega'}\left[\mathbb{E}^{\omega}f\big(V_1(U_1'(\omega'),U_2'(\omega');\omega)\big)\right]\\&=\mathbb{E}^{\omega'}\int P(1,U_1'(\omega'),du)f(u)\\&=\int\mu_1(dv)\int P(1,v,du)f(u)\\&=(\mathfrak{S}_1^*(\mu_1),f).\end{aligned}$$

Therefore, $\mathcal{D}(U_1)=\mathfrak{S}_1^*(\mu_1)$. Similar $\mathcal{D}(U_2)=\mathfrak{S}_1^*(\mu_2)$, so (U_1,U_2) is a coupling for $(\mathfrak{S}_1^*(\mu_1),\mathfrak{S}_1^*(\mu_2))$.

If we can prove that

$$\mathbb{E}^{\omega}f_K(V_1(u_1,u_2;\omega),V_2(u_1,u_2;\omega))\le\kappa f_K(u_1,u_2)\tag{4.2}$$

for all u_1, $u_2\in B$, then

$$\begin{aligned}\mathbb{E}f_K(U_1,U_2)=\mathbb{E}^{\omega'}[\mathbb{E}^{\omega}f_K(V_1(U_1',U_2';\omega),V_2(U_1',U_2';\omega))]\\\le\kappa\mathbb{E}^{\omega'}f_K(U_1',U_2')\le\kappa A'.\end{aligned}\tag{4.3}$$

So $\mathcal{K}(\mathfrak{S}_1^*(\mu_1),\mathfrak{S}_1^*(\mu_2))\le\kappa A'$ and (3.4) would follow since A' is an arbitrary number bigger than $\mathcal{K}(\mu_1,\mu_2)$. It remains to check (4.2).

Let us find $k\in[-L,\infty]$ such that $(u_1,u_2)\in Q_k$. If $k=\infty$, then $u_1=u_2$, so $V_1=V_2$ and (4.2) holds trivially. Now let $0\le k<\infty$. Then, due to (2.3),

$$\mathbb{P}\big\{(V_1,V_2)\in\textstyle\bigcup_{r\ge k+1}Q_r\big\}\ge1-C_*d_k.$$

Since $f_K\le d_{k+1}$ for $(V_1,V_2)\in\bigcup_{r\ge k+1}Q_r$ and $f_K\le\sup f_K=\widetilde{d}_{-L}$ for all (V_1,V_2), then

$$\mathbb{E}f_K(V_1,V_2)\le d_{k+1}(1-C_*d_k)+\widetilde{d}_{-L}C_*d_k.$$

As $f_K(u_1,u_2)=d_k$, then in this case

$$\frac{\mathbb{E}f_K(V_1,V_2)}{f_K(u_1,u_2)}\le\gamma_1(1-C_*d_k)+C_*\widetilde{d}_{-L}.$$

Therefore, (4.2) holds with some k-independent $\kappa<1$ if

$$C_*\widetilde{d}_{-L}\le1-\gamma_1.\tag{4.4}$$

If $-L\le k\le-1$, then $\|u_1\|,\|u_2\|\le\frac12d_k$ and $\|S(u_j)\|\le\gamma_0\frac12d_k$ for $j=1,2$. As $d_k>d_0$, $\gamma_0<\gamma_1$ and the random variable η with a positive probability is smaller than any fixed positive constant (see (1.2)), then

$$\mathbb{P}\Big\{\|V_1\|,\|V_2\|\le\frac12d_{k+1}\Big\}\ge\theta>0.\tag{4.5}$$

If $k \le -2$, then this means that

$$\mathbb{P}\{(V_1, V_2) \in \bigcup_{r \ge k+1} Q_r\} \ge \theta.$$

Since $f \le \widetilde{d}_{-L}$, then we have

$$\mathbb{E} f_K(V_1, V_2) \le \theta \widetilde{d}_{k+1} + (1-\theta)\widetilde{d}_{-L}. \tag{4.6}$$

As $f_K(u_1, u_2) = \widetilde{d}_k$, then (4.2) holds for $-L \le k \le -2$ if

$$\theta \widetilde{d}_{k+1} + (1-\theta)\widetilde{d}_{-L} = \kappa \widetilde{d}_k. \tag{4.7}$$

If $k = -1$, then for any ω from the event in the l.h.s of (4.5) we have $\|V_1\|, \|V_2\| \le \frac{1}{2} d_0$. Therefore $\|V_1 - V_2\| \le d_0$ and $(V_1, V_2) \in \bigcup_{r \ge 0} Q_r$. So the relation (4.6) still holds for $k = -1$ if we denote

$$\widetilde{d}_0 = d_0.$$

With this choice of $\widetilde{d}_0$, (4.2) holds for all negative k if so does (4.7).

The relations (4.7) are equivalent to

$$\widetilde{d}_{-L+1} = \frac{\kappa + \theta - 1}{\theta} \widetilde{d}_{-L}$$

and

$$\widetilde{d}_{-L+r} = \frac{1}{\theta}(\kappa \widetilde{d}_{-L+r-1} - (1-\theta)\widetilde{d}_{-L})$$

for $r \ge 2$. That is,

$$\widetilde{d}_{-L+r} = \frac{\widetilde{d}_{-L}}{\theta} \left[\left(\frac{\kappa}{\theta}\right)^{r-1} \left(\kappa + \theta - 1 - \frac{\theta(1-\theta)}{\kappa - \theta}\right) + \frac{\theta(1-\theta)}{\kappa - \theta} \right]$$

for $1 \le r \le L-1$.

Let us assume that $\kappa = 1 - \varepsilon$, where $0 < \varepsilon \ll 1$. Then

$$\widetilde{d}_{-L+r} = \frac{\widetilde{d}_{-L}}{\theta} \left[\left(\frac{-\varepsilon}{\theta^{r-1}(1-\theta)} + O(\varepsilon^2) \right) + \frac{\theta(1-\theta)}{1-\theta-\varepsilon} \right], \tag{4.8}$$

where $O(\varepsilon^2)$ depends on $r \le L$. Choosing $\varepsilon = \varepsilon_L$ sufficiently small, we see that the numbers $\widetilde{d}_{-L+r}$ $(0 \le r \le L)$ decay when r grow; so they satisfy all relations in (3.1) (if $\widetilde{d}_0 = d_0$).

We have seen that a function f_K, constructed using the numbers $\{\widetilde{d}_\ell\}$ as above, satisfies (4.2) and (3.1) if it satisfies (4.4) and if $\widetilde{d}_0 = d_0$. Due to (4.8), $\widetilde{d}_{-L} = \widetilde{d}_0(1 + O(\varepsilon))$. Taking $\widetilde{d}_0 = d_0$, we have $\widetilde{d}_{-L} = d_0(1+O(\varepsilon))$. Due to (2.4), $d_0 \le (1-\gamma_1)/2C_*$. So (4.4) is satisfied if ε is sufficiently small.

We have constructed constants $\widetilde{d}_k$ such that the corresponding function f_K satisfies (3.4) with some $\kappa = 1 - \varepsilon < 1$. The theorem is proved.

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