JOURNÉES ÉQUATIONS AUX DÉRIVÉES PARTIELLES

ZHONGWEI SHEN

The magnetic Schrödinger operator and reverse Hölder class

Journées Équations aux dérivées partielles (1996), p. 1-10

<http://www.numdam.org/item?id=JEDP_1996____A17_0>

The Magnetic Schrödinger Operator
And Reverse Hölder Class

Zhongwei Shen[1]

Department of Mathematics

University of Kentucky

Lexington, KY 40506

U. S. A.

Abstract: We present some recent results on the number of negative eigenvalues and eigenvalue asymptotics for magnetic Schrödinger operators. The conditions on the electric potential and magnetic field are given in terms of the reverse Hölder inequality.

1. Introduction

Consider the magnetic Schrödinger operator

$$H = H(\mathbf{a}, V) = \sum_{j=1}^{n} \left(\frac{1}{i} \frac{\partial}{\partial x_j} - a_j(x) \right)^2 + V(x) \quad \text{on } \mathbb{R}^n,\ n \geq 3 \tag{1.1}$$

where $\mathbf{a} = (a_1, \cdots, a_n) : \mathbb{R}^n \to \mathbb{R}^n$ is the magnetic potential, and $V : \mathbb{R}^n \to \mathbb{R}$ is the electric potential. We shall assume that, H admits a self-adjoint realization, which is still denoted by H, in $L^2(\mathbb{R}^n)$.

Let $N(\lambda, H)$ denote the number of eigenvalues (counting multiplicity) of H smaller than λ (or in general the dimension of the spectral projection for H corresponding to the interval $(-\infty, \lambda)$). In the case $\mathbf{a} = \mathbf{0}$, the classical theorem of Cwikel–Lieb–Rozenbljum states that

$$N(\lambda, -\Delta + V) \leq c_n \left| \left\{ (x, \xi) \in \mathbb{R}^n \times \mathbb{R}^n : \, |\xi|^2 + V(x) < \lambda \right\} \right|. \tag{1.2}$$

However, there exist some simple potentials $V(x)$ (e.g. $V(x) = x_1^2 x_2^2 \cdots x_n^2$) for which, the right hand of (1.2) is infinite and, nevertheless, $-\Delta + V(x)$ has a discrete

[1] Partially supported by the NSF

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX spectrum. Furthermore, the phase-space volume estimate fails completely when the magnetic potential $\mathbf{a}(x)$ is present and $V(x) \equiv 0$. There has been considerable interest in this kind of non-classical eigenvalue asymptotics in recent years. See e.g. [R], [Si], [F], [HM], [MN], [I]. In this note we will present some recent results on the number of negative eigenvalues and eigenvalue asymptotics for the Schrödinger operator with magnetic field in certain reverse Hölder class. Our results are closely related to the work of Fefferman and Phong [F], Helffer, Mohamed, and Nourrigat [HM], [HN1], [MN]. In particular, we generalize the Fefferman-Phong estimates on the number of negative eigenvalues for $-\Delta + V(x)$ (using minimal dyadic cubes) to the operator $H(\mathbf{a}, V)$. Our estimates incorporate the contribution from the magnetic field in an effective way. We are also able to extend the results of B. Helffer, A. Mohamed, and J. Nourrigat on the eigenvalue asymptotics to potentials with minimal smoothness assumptions. Finally, we will mention some related L^p–estimates and weak-type $(1,1)$ estimate for the magnetic Schrödinger operator.

2. The Reverse Hölder Class

Our assumptions on potentials will be given in terms of the reverse Hölder inequality.

Let $Q(x,r)$ denote the cube centered at x with side length r.

Definition 2.1. Suppose that $W \in L^p_{loc}(\mathbb{R}^n)$ $(1 < p \le \infty)$ and $W \ge 0$ a.e. on $\mathbb{R}^n$. We say $W \in (RH)_{p,\,loc}$ if there exists $C_0 \ge 1$ such that

$$\left(\frac{1}{r^n}\int_{Q(x,r)} W^p(y)dy\right)^{1/p} \le C_0 \cdot \frac{1}{r^n}\int_{Q(x,r)} W(y)dy \tag{2.2}$$

for every $x \in \mathbb{R}^n$ and $0 < r \le 1$. If (2.2) holds for $0 < r < \infty$, we say $W \in (RH)_p$.

The reverse Hölder class $(RH)_p$ was introduced by Gehring and Muckenhoupt in the study of quasi-conformal mapping and weighted norm inequalities, respectively. It has been studied extensively in harmonic analysis. See [St].

Example 2.3. If $\alpha > 0$ and $P(x)$ is a polynomial of degree k, then $W(x) = |P(x)|^\alpha \in (RH)_p$ for any $p > 1$ with a constant $C_0 = C_0(n, \alpha, k)$.

Example 2.4. $W(x) = e^{|x|}$ is in $(RH)_{p,\,loc}$, but not in $(RH)_p$ for any $p > 1$.

Definition 2.5. For a nonnegative function W, the auxilary function $m(x, W)$ is defined by

$$\frac{1}{m(x,W)} = \sup\left\{ r > 0 : \frac{1}{r^{n-2}} \int_{Q(x,r)} W(y)dy \le 1 \right\}.$$

The function $m(x, W)$, which is closely related to the uncertainty principle, plays a very important role. The definition of $m(x, W)$ generalizes the earlier version of a very useful auxiliary function for polynomial potentials. Indeed, if $W = |P(x)|$ and $P(x)$ is a polynomial of degree k, then

$$m(x, W) \approx \sum_{|\beta| \le k} |\partial_x^\beta P(x)|^{\frac{1}{|\beta|+2}}. \tag{2.6}$$

The following proposition summarizes the basic properties of $m(x, W)$ when $W \in (RH)_{n/2}$ [Sh1].

Proposition 2.7. *Suppose* $W \in (RH)_{n/2}$. *Then there exist* $C > 0$, $c > 0$, *and* $k_0 > 0$ *such that*

(a) $$m(x,W) \approx m(y,W) \quad \text{if } |x-y| \le \frac{C}{m(x,W)},$$

(b) $$m(y,W) \le C\left\{1 + |x-y| m(x,W)\right\}^{k_0} m(x,W),$$

(c) $$m(y,W) \ge \frac{c\, m(x,W)}{\{1+|x-y|m(x,W)\}^{k_0/(k_0+1)}}.$$

Similar properties hold if we assume $W \in (RH)_{n/2,\,loc}$ and restrict x, y to the case $|x - y| \le 1$ [Sh3].

3. The Number of Negative Eigenvalues

Using a sharper form of the uncertainty principle, C. Fefferman and D. H. Phong were able to refine the classical estimate (1.2). In [F], it was shown that, for $p > 1$ and $\lambda \leq 0$, $N(\lambda, -\Delta + V)$ is bounded by $C_n \cdot N_0$, where N_0 is the number of minimal (disjoint) dyadic cubes which satisfy

$$\ell(Q)^2 \left(\frac{1}{|Q|} \int_Q |V(x)|^p dx \right)^{1/p} \geq c > 0, \quad \ell(Q) < \frac{1}{\sqrt{|\lambda|}}, \tag{3.1}$$

$\ell(Q)$ denotes the side length of cube Q, and c depends on n and p.

In this section we generalize the Fefferman–Phong estimate to the magnetic Schrödinger operator under certain conditions on the magnetic field $\mathbf{B}(x)$. The conditions on $\mathbf{B}$ in particular are satisfied if the magnetic potentials $a_j(x)$, $j = 1, 2, \ldots, n$ are polynomials. More importantly, our estimates incorporate the contribution from the magnetic field.

Let $\mathbf{B}(x) = \text{curl } \mathbf{a}(x) = (b_{jk}(x))_{1 \leq j,k \leq n}$ be the magnetic field generated by $\mathbf{a}(x)$ where

$$b_{jk}(x) = \frac{\partial a_j}{\partial x_k} - \frac{\partial a_k}{\partial x_j}. \tag{3.2}$$

Theorem 3.3. [Sh5] *Let $n \geq 3$. Suppose $\mathbf{a} \in C^2(\mathbb{R}^n)$, $V \in L^p_{loc}(\mathbb{R}^n)$ for some $p > 1$. Also assume that $|\mathbf{B}| \in (RH)_{n/2}$ and*

$$|\nabla \mathbf{B}(x)| \leq C_1 \left\{ m(x, |\mathbf{B}|) \right\}^3 \tag{3.4}$$

where $|\mathbf{B}| = |\mathbf{B}(x)| = \sum_{j,k} |b_{jk}(x)|$. Then, there exist $C = C(n) > 0$ and $c = c(C_0, C_1, n, p) > 0$, such that, for $\lambda \leq 0$, $N(\lambda, H)$ is bounded by $C \cdot N_0$ where N_0 is the number of minimal (disjoint) dyadic cubes Q which satisfy

$$\ell(Q)^2 \left(\frac{1}{|Q|} \int_Q |V(x)|^p dx \right)^{1/p} \geq c, \quad \ell(Q) < \frac{1}{\sqrt{|\lambda|}}, \tag{3.5}$$

and

$$(3.6) \qquad \ell(Q)^2 \left(\frac{1}{|Q|} \int_Q |\mathbf{B}(x)|^2 dx \right)^{1/2} \leq 1.$$

Remark 3.7. Note that the conditions $|\mathbf{B}| \in (RH)_{n/2}$ and $|\nabla \mathbf{B}(x)| \leq C \{m(x, |\mathbf{B}|)\}^3$ in Theorem 3.3 are dilation invariant. Roughly speaking, these two conditions mean that the values of $|\mathbf{B}|$ do not fluctuate too much on the average and $|\nabla \mathbf{B}|$ is uniformly bounded in the scale $\{m(x, |\mathbf{B}|)\}^{-1}$. It follows easily from (2.6) that the hypothesis of Theorem 3.3 is satisfied if the magnetic potentials $a_j(x)$ are polynomials. Moreover, in this case, the constants C_0, C_1 depend only on n and the degrees of $a_j(x)$.

Remark 3.8. The condition (3.6) in Theorem 3.3 may be replaced by

$$(3.9) \qquad \ell(Q)^2 \left(\frac{1}{|Q|} \int_Q |\mathbf{B}(x)|^q dx \right)^{1/q} \leq 1$$

where $0 < q \leq \infty$ [Sh5].

Corollary 3.10. [Sh5] *Under the same assumption as in Theorem 3.3, we have*

$$(3.11) \qquad N(0, H) \leq C \int_{\{x \in \mathbb{R}^n : V(x) < 0\}} \frac{|V(x)|^p}{\{m(x, |\mathbf{B}|)\}^{2p-n}} dx$$

for $p \geq n/2$, where C depends on n, p, C_0 and C_1.

In the case $p = n/2$, this is the classical Cwikel-Lieb-Rozenbljum estimate.

The following lower bound estimate suggests that the upper bound in Theorem 3.3 is almost optimal.

Theorem 3.12. [Sh5] *Suppose* $\mathbf{a} \in C^1(\mathbb{R}^n)$, $V \in L^1_{loc}(\mathbb{R}^n)$ *and* $V \leq 0$ *a.e. on* $\mathbb{R}^n$. *Then, there exists $C_2 > 0$ depending only on n, such that, if there exists a collection of cubes $\{Q_k, k = 1, 2, \ldots, N_0\}$, whose double are pointwise disjoint, with the properties*

$$\ell(Q)^2 \left(\frac{1}{|Q|} \int_Q |V| dx \right) \geq C_2, \quad \ell(Q) < \frac{1}{\sqrt{|\lambda|}},$$

and

$$\ell(Q)^2 \left(\frac{1}{|Q|} \int_{2Q} |\mathbf{B}|^2 dx \right)^{1/2} \leq 1,$$

then

$$N(\lambda, H) \geq N_0.$$

4. The Eigenvalue Asymptotics

Theorem 4.1. [Sh3] *Suppose* $\mathbf{a} \in C^2(\mathbb{R}^n)$, $V \in L^{n/2}_{loc}(\mathbb{R}^n)$ *and* $V \geq 0$ *a.e. on* $\mathbb{R}^n$, $n \geq 3$. *Also assume that*

$$\begin{cases} |\mathbf{B}| + V + 1 \in (RH)_{n/2,\, loc}, \\ |\nabla \mathbf{B}(x)| \leq C_1 \{1 + m(x, |\mathbf{B}| + V)\}^3 . \end{cases}$$

Then, there exist constants $C = C(n, C_0, C_1) > 0$ *and* $c = c(n, C_0, C_1) > 0$, *such that, for* $\lambda \geq C$,

$$N(\lambda, H(\mathbf{a}, V)) \leq C | \left\{ (x, \xi) : |\xi|^2 + \{c\, m(x, |\mathbf{B}| + V)\}^2 < \lambda \right\} |,$$

and

$$N(\lambda, H(\mathbf{a}, V)) \geq c | \left\{ (x, \xi) : |\xi|^2 + \{C\, m(x, |\mathbf{B}| + V)\}^2 < \lambda \right\} |,$$

where $(x, \xi) \in \mathbb{R}^n \times \mathbb{R}^n$.

Theorem 4.1, which generalizes a result by Helffer, Mohamed, and Nourrigat [HM] [MN], allows one to estimate the leading power of $N(\lambda, H(\mathbf{a}, V))$ in many cases for degenerate potentials $V(x)$, as well as degenerate magnetic fields $\mathbf{B}(x)$. Indeed, it follows easily from Theorem 4.1 that

$$N(\lambda, H) \leq C\, \lambda^{n/2} | \left\{ x \in \mathbb{R}^n : \ m(x, |\mathbf{B}| + V) \leq C\sqrt{\lambda} \right\} |,$$
$$N(\lambda, H) \geq c\, \lambda^{n/2} | \left\{ x \in \mathbb{R}^n : \ m(x, |\mathbf{B}| + V) \leq c\sqrt{\lambda} \right\} |.$$

Corollary 4.2. *Suppose* $\mathbf{a}(x)$ *and* $V(x)$ *satisfy the same hypothesis of Theorem 4.1. Then* $H(\mathbf{a}, V)$ *has a discrete spectrum if and only if*

$$m(x, |\mathbf{B}| + V) \to \infty \quad \text{as } |x| \to \infty.$$

5. The Uncertainty Principle

The proof of Theorem 3.1 and Theorem 4.1 relies on a new form of the uncertainty principle.

Theorem 5.1. [Sh3] *Suppose* $\mathbf{a} \in C^2(\mathbb{R}^n)$, $V \in L^{n/2}_{loc}(\mathbb{R}^n)$ *and* $V(x) \geq 0$. *Also assume that*

$$\begin{cases} |\mathbf{B}| + V + 1 \in (RH)_{n/2,\, loc} \\ |\nabla \mathbf{B}(x)| \leq C_1 \left\{ m(x, |\mathbf{B}| + V + 1) \right\}^3 . \end{cases}$$

Then, for $u \in C_0^1(\mathbb{R}^n)$,

$$\int_{\mathbb{R}^n} |m(x, |\mathbf{B}| + V + 1)\, u(x)|^2 dx$$
$$\leq C \left\{ \sum_{j=1}^{n} \int_{\mathbb{R}^n} \left|\left(\frac{1}{i}\frac{\partial}{\partial x_j} - a_j(x)\right) u\right|^2 dx + \int_{\mathbb{R}^n} (V(x)+1)|u|^2 dx \right\}.$$

Corollary 5.2. [Sh5] *Suppose* $\mathbf{a} \in C^2(\mathbb{R}^n)$. *Also assume that* $\mathbf{B} \in (RH)_{n/2}$ *and*

$$|\nabla \mathbf{B}(x)| \leq C_1 \left\{ m(x, |\mathbf{B}|) \right\}^3 .$$

Then

$$\int_{\mathbb{R}^n} |m(x, |\mathbf{B}|)\, u(x)|^2 dx \leq C \sum_{j=1}^{n} \int_{\mathbb{R}^n} \left|\left(\frac{1}{i}\frac{\partial}{\partial x_j} - a_j(x)\right) u\right|^2 dx.$$

The estimate in Corollary 5.2 implies that the operator $H(\mathbf{a}, 0)$ is bounded from below by $\{m(x, |\mathbf{B}|)\}^2$. Using this lower bound, we may deduce the following decay estimate

$$|\Gamma_\lambda(x,y)| \leq \frac{C_k}{\{1 + |x-y| m(x, |\mathbf{B}| + |\lambda|)\}^k} \cdot \frac{1}{|x-y|^{n-2}} \tag{5.3}$$

where $\Gamma_\lambda(x,y)$ denotes the kernel function of the operator $(H(\mathbf{a},0) + |\lambda|)^{-1}$ and k is any positive integer.

To prove Theorem 3.3, we follow the approach of Fefferman and Phong [F]. Also see [KS]. The key step, which requires the systematic control over the magnetic field $\mathbf{B}$, is to establish the following trace inequality:

$$\int_{\mathbb{R}^n} |V|\, |g|^2 dx \le C \cdot M_p \left\{ \sum_{j=1}^n \int_{\mathbb{R}^n} |(\frac{1}{i}\frac{\partial}{\partial x_j} - a_j) g|^2 dx + |\lambda| \int_{\mathbb{R}^n} |g|^2 dx \right\} \tag{5.4}$$

where

$$M_p = \sup_Q \ell(Q)^2 \left(\frac{1}{|Q|} \int_Q |V|^p dx \right)^{1/p} \tag{5.5}$$

and the supremum is over all dyadic cubes Q satisfying

$$\ell(Q) < \inf_{x \in Q} \frac{\alpha}{m(x, |\mathbf{B}| + |\lambda|)}. \tag{5.6}$$

Note that (5.4) is equivalent to

$$\int_{\mathbb{R}^n} |V|\, |(H(\mathbf{a}, 0) + |\lambda|)^{-1/2} f|^2 dx \le C \cdot M_p \int_{\mathbb{R}^n} |f|^2 dx. \tag{5.7}$$

Let $K_\lambda(x, y)$ denote the kernel function of the operator $(H(\mathbf{a}, 0) + |\lambda|)^{-1/2}$. It follows from (5.3) that, for any $k > 0$,

$$|K_\lambda(x, y)| \le \frac{C_k}{\{1 + |x - y| m(x, |\mathbf{B}| + |\lambda|)\}^k} \cdot \frac{1}{|x - y|^{n-1}}. \tag{5.8}$$

The proof of the trace inequality (5.4) is based on (5.8) and techniques from harmonic analysis. See [Sh5].

6. The L^p Estimates

In this section we give the L^p and weak-type $(1,1)$ estimates for the magnetic Schrödinger operator (1.1). For $-\Delta + V(x)$, similar results can be found in [HN1] [Gu] [Z] [Sh1]. For the Schrödinger operator with magnetic field, the only known result is an L^2-estimate given by Guibourg [Gu] for potentials which behave like polynomials.

Let $L_j = \frac{1}{i}\frac{\partial}{\partial x_j} - a_j(x)$.

Theorem 6.1. [Sh4] Suppose $\mathbf{a} \in C^2(\mathbb{R}^n)$, $V \in L^\infty_{loc}(\mathbb{R}^n)$ and $V \geq 0$ a.e. on $\mathbb{R}^n$, $n \geq 3$. Also assume that

$$\begin{cases} |\mathbf{B}| + V + 1 \in (RH)_{n/2,\, loc}, \\ V(x) \leq C_1 \{m(x, |\mathbf{B}| + V + 1)\}^2, \\ |\nabla \mathbf{B}(x)| \leq C_2 \{m(x, |\mathbf{B}| + V + 1)\}^3. \end{cases}$$

Then, for $1 < p < \infty$,

$$\sum_{1 \leq j,k \leq n} \|L_j L_k(f)\|_p \leq C\{\|H(\mathbf{a}, V)f\|_p + \|f\|_p\}$$

for any $f \in C_0^\infty(\mathbb{R}^n)$. We also have the weak-type $(1,1)$ estimate

$$|\{x \in \mathbb{R}^n : \sum_{1 \leq j,k \leq n} |L_j L_k(f)(x)| > \lambda\}| \leq \frac{C}{\lambda}\{\|H(\mathbf{a}, V)f\|_1 + \|f\|_1\}.$$

References

[AHS] J. Avron, I. Herbst and B. Simon, *Schrödinger Operators with Magnetic Fields. I. General Interaction*, Duke Math. J. **45(4)** (1978), 847–883.

[F] C. Fefferman, *The Uncertainty Principle*, Bull. Amer. Math. Soc. **9** (1983), 129–206.

[Gu] D. Guibourg, *Inégalités Maximales pour L'Opérateur de Schrödinger*, Ph.D. Thesis, Université de Rennes-I (1992.).

[Gur] D. Gurarie, *Nonclassical Eigenvalue Asymptotics for Operators of Schrödinger Type*, Bull. Amer. Math. Soc. **15(2)** (1986), 233–237.

[H] B. Helffer, *Semi-Classical Analysis for the Schrödinger Operator and Applications, Lectures Notes in Math.*, vol. 1336, Springer–Verlag, 1988.

[HM] B. Helffer and A. Mohamed, *Caractérisation du spectre essentiel de lópérateur de Schrödinger avec un champ magnétique*, Ann. Inst. Fourier **38** (1988), 95–112.

[HN1] B. Helffer and J. Nourrigat, *Hypoellipticité maximale pour des opérateurs polynômes de champs de vecteurs*, Progress in Math. 58, Birkhäuser, Boston (1985).

[HN2] B. Helffer and J. Nourrigat, *Decrossance a l'infini des fonctions propres de l'opérateur de Schrödinger avec champ electromagnétique polynomial*, J. Anal. Math. **58** (1992), 263–275.

[I] A. Iwatsuka, *Magnetic Schrödinger Operators with Compact Resolvent*, J. Math. Kyoto Univ. **26–3** (1986), 357–374.

[KS] R. Kerman and E. Sawyer, *The Trace Inequality and Eigenvalue Estimates for Schrödinger Operators*, Ann. Inst. Fourier(Grenoble) **36(4)** (1986), 207–228.

[MN] A. Mohamed and J. Nourrigat, *Encadrement du $N(\lambda)$ pour un opérator de Schrödinger avec un champ magnétique et un potentiel électrique*, J. Math. Pures Appl. **70** (1991), 87–99.

[R] D. Robert, *Comportement asymptotique des valurs propres d'opérateurs du type Schrödinger á potentiel "dégénéré"*, J. Math. Pures Appl. **61(9)** (1982), 275-300.

[Sh1] Z. Shen, *L^p Estimates for Schrödinger Operators with Certain Potentials*, Ann. Inst. Fourier (Grenoble) **45(2)** (1995), 513–546.

[Sh2] Z. Shen, *On the Eigenvalue Asymptotics of Schrödinger Operators*, preprint (1995).

[Sh3] Z. Shen, *Eigenvalue Asymptotics and Exponential Decay of Eigenfunctions for Schrödinger Operators with Magnetic Fields*, Trans. Amer. Math. Soc. (to appear).

[Sh4] Z. Shen, *Estimates in L^p for Magnetic Schrödinger Operators*, preprint (1995).

[Sh5] Z. Shen, *On the Number of Negative Eigenvalues for a Schrödinger Operator with Magnetic Field*, Comm. Math. Phys. (to appear).

[Si] B. Simon, *Nonclassical Eigenvalue Asymptotics*, J. Funct. Anal. **53** (1983), 84–98.

[So] M. Solomjak, *Spectral Asymptotics of Schrödinger Operators with Non–regular Homogeneous Potential*, Math. USSR Sb. **55** (1986), 19–38.

[St] E. Stein, *Harmonic Analysis: Real-Variable Method, Orthogonality, and Oscillatory Integrals*, Princeton Univ. Press, 1993.

[Z] J. Zhong, *Harmonic Analysis for Some Schrödinger Type Operators*, Ph.D. Thesis, Princeton University, 1993.

E-mail: shenz@ms.uky.edu