

AN ESTIMATE ON THE HESSIAN OF THE HEAT KERNEL

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ABSTRACT. Let M be a compact, connected Riemannian manifold, and let $p_t(x, y)$ denote the fundamental solution to Cauchy initial value problem for the heat equation $\frac{\partial u}{\partial t} = \frac{1}{2}\Delta u$, where Δ is the Levi-Civita Laplacian. The purpose of this note is to describe the behavior of the Hessian of $\log p_T(\cdot, y)$ for small $T > 0$.

Emphasis is given to the difference between what happens outside, where the behavior is like $\frac{1}{T}$, as opposed to at the cut locus, where it is like $\frac{1}{T^2}$.

§0: INTRODUCTION

Let M be a compact, connected, d -dimensional Riemannian manifold, denote by $\mathcal{O}(M)$ with fiber map $\pi : \mathcal{O}(M) \rightarrow M$ the associated bundle of orthonormal frames ϵ , and use the Levi-Civita connection to determine the horizontal subspace $H_\epsilon(\mathcal{O}(M))$ at each $\epsilon \in \mathcal{O}(M)$. Next, given $\mathbf{v} \in \mathbb{R}^d$, let $\mathfrak{E}(\mathbf{v})$ be the *basic vector field* on $\mathcal{O}(M)$ determined by properties that

$$\mathfrak{E}(\mathbf{v})_\epsilon \in H_\epsilon(\mathcal{O}(M)) \quad \text{and} \quad d\pi \mathfrak{E}(\mathbf{v})_\epsilon = \epsilon \mathbf{v} \quad \text{for all } \epsilon \in \mathcal{O}(M).$$

(Here, and whenever convenient, we think of ϵ as a isometry from \mathbb{R}^d onto $T_{\pi(\epsilon)}(M)$.) In particular, if $\{\mathbf{e}_1, \dots, \mathbf{e}_d\}$ is the standard orthonormal basis in \mathbb{R}^d , then we set $\mathfrak{E}_k(\epsilon) = \mathfrak{E}(\mathbf{e}_k)_\epsilon$. If, for $\mathcal{O} \in \mathcal{O}(d)$ (the orthogonal group on \mathbb{R}^d) $R_{\mathcal{O}} : \mathcal{O}(M) \rightarrow \mathcal{O}(M)$ is defined so that

$$R_{\mathcal{O}} \epsilon \mathbf{v} = \epsilon \mathcal{O} \mathbf{v}, \quad \epsilon \in \mathcal{O}(M) \text{ and } \mathbf{v} \in \mathbb{R}^d,$$

then it easy to check that

$$(0.1) \quad dR_{\mathcal{O}} \mathfrak{E}(\mathbf{v})_\epsilon = \mathfrak{E}(\mathcal{O}^\top \mathbf{v})_{R_{\mathcal{O}} \epsilon}, \quad \epsilon \in \mathcal{O}(M) \text{ and } \mathbf{v} \in \mathbb{R}^d.$$

Given a smooth function F on $\mathcal{O}(M)$, we define $\nabla F : \mathcal{O}(M) \rightarrow \mathbb{R}^d$, $\text{Hess}(F) : \mathcal{O}(M) \rightarrow \mathbb{R}^d \otimes \mathbb{R}^d$, and $\Delta F : \mathcal{O}(M) \rightarrow \mathbb{R}$ by

$$(0.2) \quad \nabla F = \sum_1^d \mathfrak{E}_k F \mathbf{e}_k, \quad \text{Hess}(F) = ((\mathfrak{E}_k \circ \mathfrak{E}_\ell F))_{1 \leq k, \ell \leq d}$$

$$\text{and} \quad \Delta F = \sum_1^d \mathfrak{E}_k^2 F.$$

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In particular, when f is a smooth function on M , we set

$$\nabla f \equiv \nabla(f \circ \pi), \quad \text{Hess}(f) \equiv \text{Hess}(f \circ \pi), \quad \text{and} \quad \Delta f \equiv \Delta(f \circ \pi).$$

Starting from (0.1), it is an easy matter to check that

$$\begin{aligned} (\nabla f) \circ R_{\mathcal{O}} &= \mathcal{O}^{\top} \nabla f, & (\text{Hess}(f)) \circ R_{\mathcal{O}} &= \mathcal{O}^{\top} \text{Hess}(f) \mathcal{O}, \\ & \text{and} & (\Delta f) \circ R_{\mathcal{O}} &= \Delta f. \end{aligned}$$

Hence, $|\nabla f|$, $\|\text{Hess}(f)\|_{\text{H.S.}}$ (the Hilbert–Schmidt norm), and Δf are all well-defined on M . In fact, Δf is precisely the action of the Levi–Civita Laplacian on f .

Now consider Cauchy initial value for the heat equation

$$\frac{\partial u}{\partial t} = \frac{1}{2} \Delta u, \quad t \in (0, \infty) \quad \text{with} \quad \lim_{t \searrow 0} u(t, x) = f(x), \quad x \in M.$$

By standard elliptic regularity theory, one knows that there is a unique, smooth function $(t, x, y) \in (0, \infty) \times M \times M \mapsto p_t(x, y) \in (0, \infty)$ such that

$$u(t, x) = \int_M f(y) p_t(x, y) \lambda_M(dy), \quad (t, x) \in (0, \infty) \times M \text{ and } f \in C(M; \mathbb{R}),$$

where λ_M denotes the normalized Riemann measure on M . Moreover, because Δ is essentially self-adjoint in $L^2(\lambda_M)$, $p_t(x, y) = p_t(y, x)$.

§1: THE RESULTS

We begin by considering the logarithmic gradient $\nabla \log p_T(\cdot, y)$, for which our initial result depends only on the dimension d and the lower bound

$$(1.1) \quad \alpha \equiv \min_{\mathbf{e} \in \mathcal{O}(M)} \min_{\mathbf{v} \in S^{d-1}} (\mathbf{v}, \text{Ric}(\mathbf{e})\mathbf{v})_{\mathbb{R}^d}$$

for the Ricci curvature. One (cf. [SZ]) can then show that there is a

$$(1.2) \quad C(d, \alpha) < i\infty \text{ such that, for each } \epsilon \in (0, 1),$$

$$|\nabla \log p_T(\cdot, y)|(x) \leq \frac{((1 + \epsilon)e^{\alpha T})^{\frac{1}{2}} \rho(x, y)}{T} + \frac{C(d, \alpha)}{(\epsilon T)^{\frac{1}{2}}}, \quad (T, x, y) \in (0, 1] \times M^2,$$

where we have introduced $\rho(x, y)$ to denote the Riemannian distance between x and y .

Notice that the preceding result does not feel the cut locus. To get a result which does, we look at what happens asymptotically as $T \searrow 0$. What one finds (cf. the first part of Theorem 3.12 in [KS]) is that

$$(1.3) \quad \begin{aligned} & y \text{ outside the cut locus of } x \equiv \pi(\mathbf{e}) \implies \\ & \lim_{T \searrow 0} T [\nabla \log P_T(\cdot, y)](\geq) = \mathbf{v}(\mathbf{e}, y), \end{aligned}$$

where $\mathbf{v}(\boldsymbol{\epsilon}, y)$ is the element of \mathbb{R}^d which is determined by the requirement that the path $f \in C^1([0, 1]; \mathcal{O}(M))$ satisfying

$$(1.4) \quad f(0) = \boldsymbol{\epsilon} \text{ and } \dot{f}(t) = \mathfrak{E}(\mathbf{v}(\boldsymbol{\epsilon}, y))_{\boldsymbol{\epsilon}(t)}$$

is the horizontal lift to $\boldsymbol{\epsilon}$ of the (unique) minimal geodesic going from x to y . When y is at the cut locus of x , one should not expect (1.3) to hold. In fact, take $S(x, y)$ in $T_x(M)$ to be the set of initial directions in which minimal geodesics from x to y can proceed. When $S(x, y)$ forms a non-trivial differentiable submanifold, then one can use the second part of Theorem 3.12 in [KS] to see that the limit on the left side of (1.3) exists and is a non-trivial convex combination of elements of $\boldsymbol{\epsilon}^{-1}(S(x, y))$. In particular, since all elements of have the same length, this limit has length strictly less than $\rho(x, y)$ in this case. For example, when M is the circle centered at the origin in \mathbb{R}^2 with unit circumference,

$$(1.5) \quad p_T(\theta, \frac{1}{2}) = (2\pi T)^{-\frac{1}{2}} \sum_{m \in \mathbb{Z}} \exp\left(-\frac{(\theta - \frac{1}{2} - m)^2}{2T}\right),$$

and so it is clear that

$$\lim_{T \searrow 0} T [\nabla \log p_T(\cdot, \frac{1}{2})](0) = 0.$$

The analysis of the Hessian of $\log p_T(\cdot, y)$ is more challenging. What it leads to is a general estimate (cf. [S]) of the form

$$(1.6) \quad -\frac{C}{T} \leq [\text{Hess} \log p_T(\cdot, y)](\boldsymbol{\epsilon}) \leq C \left(\frac{1}{T} + \frac{\rho(x, y)^2}{T^2} \right)$$

for $\boldsymbol{\epsilon} \in \pi^{-1}(x)$ and $(T, x, y) \in (0, 1] \times M^2$.

Unlike the constant in (1.2), the C in (1.6) depends on more than the lower bound α in (1.2). In fact, asymptotic analysis based on [KS] gives

y outside the cut locus of \implies

$$(1.7) \quad \lim_{T \searrow 0} T [\text{Hess} \log p_T(\cdot, y)](\boldsymbol{\epsilon}) = -\mathbf{I} + \int_0^1 (1-t)^2 \text{Sec}(f(t), \mathbf{v}(\boldsymbol{\epsilon}, y)) dt,$$

where $\mathbf{v}(\boldsymbol{\epsilon}, y) \in \mathbb{R}^d$ and $f \in C^1([0, 1]; \mathcal{O}(M))$ are defined as above (cf. (1.4)) and $\text{Sec}: \mathcal{O}(M) \times \mathbb{R}^d \longmapsto \mathbb{R}^d \otimes \mathbb{R}^d$ is the (unnormalized) sectional curvature given by

$$(\xi, \text{Sec}(\mathbf{g}, \mathbf{v})\eta)_{\mathbb{R}^d} = (\text{Riem}_{\mathbf{g}}(\xi, \mathbf{v})\eta, \mathbf{v})_{\mathbb{R}^d}.$$

On the other hand, when y is at the cut locus of x and the set $S(x, y)$

has the sort of structure described in the preceding paragraph, then one can show that

$$\lim_{T \searrow 0} T^2 [\text{Hess} \log p_T(\cdot, y)](\boldsymbol{\epsilon}) \text{ exists and is strictly positive definite.}$$

For example, in the case of the circle considered above,

$$\lim_{T \searrow 0} T^2 [\text{Hess } \log p_T(\cdot, \frac{1}{2})] (0) = \frac{1}{4}.$$

The proofs of these results are based on probabilistic representations of $p_T(\cdot, y)$ and its derivatives in terms of the Brownian motion on M (cf. (2.2) and (2.12) in [S]).

Remark: Because, by an old result of Varadhan's, one knows that

$$\lim_{T \searrow 0} T \log p_T(x, y) = \frac{\rho(x, y)^2}{2} \text{ for all } x, y \in M,$$

the expression on the right hand side of (1.7) must equal the Hessian of $\frac{1}{2}\rho(\cdot, y)^2$. However, to date, the author has found no corroboration in differential geometry texts.

REFERENCES

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