

# Recent Progress on the Blow-up Problem for Zakharov Equations

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In this paper, we present recent progress for the blow-up problem for Zakharov equations.

More precisely, we consider Zakharov equations

$$(I) \quad \begin{aligned} i\partial u/\partial t &= -\Delta u + nu \\ \partial n/\partial t &= -\nabla \cdot v \\ c_0^{-2}\partial v/\partial t &= -\nabla n - \nabla |u|^2 \end{aligned}$$

with initial data  $(u(-1), n(-1), v(-1)) = (u_0, n_0, v_0)$

where  $u : \mathbb{R}^2 \rightarrow \mathbb{C}$ ,  $n : \mathbb{R}^2 \rightarrow \mathbb{R}$ ,  $v : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ ,

and related equations which are

the cubic nonlinear Schrödinger equation

$$(II) \quad i\partial u/\partial t = -\Delta u - |u|^2 u$$

with initial data  $u(-1) = u_0$

where  $u : \mathbb{R}^2 \rightarrow \mathbb{C}$ ,

and the Elliptic equation associated with (II)

$$(III) \quad u = \Delta u + |u|^2 u$$

where  $u : \mathbb{R}^2 \rightarrow \mathbb{C}$ .

1 ) The local Cauchy theory for equations (I),(II).

We are interested to find a space  $H$  for equation (I) or (II) such that there is a unique solution of the equation on  $[0, T)$  and we have the following  $T=+\infty$  or  $T<+\infty$  and  $\|u(t)\|_H \rightarrow +\infty$  as  $t$  goes to  $T$ .

i) Case of the nonlinear Schrödinger equation (II)

The case of the cubic nonlinear Schrödinger equation is now well-understood. A local (in time) Cauchy theory can be done in various natural space  $H^1, H^s, L^2$  (see [GV],[K],[CaW],[Bo1]). Moreover, one can show that the blow-up time does not depend on the Cauchy space and in fact

we have at the blow-up a concentration phenomenon in  $L^2$ .

In [MT] (see also [W2], [GIM2]), it is proved the following. Let  $u(t)$  a blow-up solution (and  $T$  its blow-up time), there is then  $x(t)$  such that for all  $R>0$ ,  $\liminf_{t \rightarrow T} \int_{|x-x(t)| \leq R} |u(t)|^2 dx \geq a > 0$  where  $a$  is an universal constant ( $a=|Q|_{L^2}^2$  where  $Q$  will be defined in subsection 3).

In addition, we have the following conserved quantities for all  $t$ ,

$$\|u(t)\|_{L^2} = \|u_0\|_{L^2},$$

$$E(u(t)) = E(u_0) \text{ where } E(u) = 1/2 \int |\nabla u|^2 - 1/4 \int |u|^4.$$

ii) Case of Zakharov equations (I).

A local (in time) Cauchy theory can not be done up to now in the energy space  $H_1 = \{(u, n, v) \in H^1 \times L^2 \times L^2\}$  for a general initial data. The result is proved for the space  $H_2 = \{(u, n, v) \in H^2 \times H^1 \times H^1\}$  (see for exemple [OT2], [KePVg], [Bo2] and the references therein).

Moreover, one can show that we have at the blow-up time again the same concentration phenomenon in  $L^2$ . Indeed, let  $(u(t), n(t), v(t))$  a blow-up solution (and  $T$  its blow-up time), there is then  $x(t)$  such that for all  $R>0$ ,  $\liminf_{t \rightarrow T} \int_{|x-x(t)| \leq R} |u(t)|^2 dx \geq |Q|_{L^2}^2$  where  $Q$  will be defined in subsection 3.

In addition, we have the following conserved quantities for all  $t$ ,

$$\|u(t)\|_{L^2} = \|u_0\|_{L^2},$$

$$H(u(t), n(t), v(t)) = H(u_0, n_0, v_0) \text{ where } H(u) = \int |\nabla u|^2 + n|u|^2 + n^2/2 + |v|^2/2c_0.$$

iii) Blow-up problem

We are now interested in the case  $T < +\infty$ , that is the case of a blow-up solution (or equivalently a singular solution) for equation (I) or (II). Most of the results can be extend in dimension  $N \geq 1$  in the case of a critical power for the nonlinear Schrödinger equation. Part of the results for the Zakharov equation can be extend to the dimension 3 (only dimensions 2,3 are relevant).

## 2) Elementary relations between equations (I)-(II)-(III)

i) Limit as  $c_0$  goes to infinity.

We can easily see that as  $c_0$  goes to infinity, the wave part of equation (I) give formally

$$\nabla(n + |u|^2) = 0,$$

or equivalently

$$n + |u|^2 = 0.$$

Thus equation (I) transform in equation (II) as  $c_0$  goes to infinity.

If the initial data are compatible, this result of convergence has been rigourously proved by several authors ([AA2], [OT1], [KePVe]) when the limit solution  $u(t)$  (of equation (II)) is regular. Near the blow-up time, we do not have convergence results and in some sense we can not expect some. For example, in [GIM2], there is the case of a blow-up solution of equation (II) with initial data  $u_0$  such that for all finite  $c_0$  and all  $n_0, v_0$  the solution of (I)  $(u, n, v)(t)$  is globally defined in time. Therefore, in some sense at the singularity, equation (I) when  $c_0$  is large, can

not be consider as a perturbation of equation (II).

ii) Periodic solutions of (I),(II)

By direct calculation, we can check that if  $w(x)$  is a solution of equation (III) then

-  $u(t,x) = e^{it} w(x)$  is a periodic solution of equation (II)

-  $(u(t,x),n(t,x),v(t,x)) = (e^{it} w(x), -|w(x)|^2, 0)$  is also a periodic solution of equation (I).

iii) Conformally self-similar blowing-up solution

For this power in two dimension, the nonlinear Schrödinger equation has one more invariance : if  $u(t,x)$  is a solution of equation (II) then

$$1/t u(1/t, x/t) \exp(i|x|^2/4t)$$

is also a solution of equation (II).

In particular, if  $w(x)$  is a real solution of the equation (III), then

$$1/t w(x/t) \exp(-i/t + i|x|^2/4t)$$

is also a solution of equation (II) which blow-up at  $T = 0$ . We then obtain explicit blow-up solutions of equation (II).

Unfortunately, such invariance does not exist for the Zakharov equation. In particular, there is no direct way to obtain explicit blow-up solutions of Zakharov equations.

### 3) On minimal solutions of (III)

In this section, we recall briefly some results on the elliptic equation (III). From [BeL],[St] it is now classical that equation (III) have infinitely many solutions in  $H^1$  (up to the invariance of the equation).

Let us defined the unique positive radially symmetric solution of equation (III) (see [Kw] for uniqueness). We have in fact that the solution  $w=0$  is isolated in the set of solution in  $L^2$ . More preciselly,

i) Assume that  $w(x)$  is a nonzero solution of equation (III) then  $\|w\|_{L^2} \geq \|Q\|_{L^2}$ .

ii) Moreover, we have the following carracterisation of the minimal solution (or ground state) of equation (III). Assume that  $w$  is a nonzero solution of equation (III) and  $\|w\|_{L^2} = \|Q\|_{L^2}$  then up to the invariance of the equation  $w = Q$  (that is there exist  $x', \omega, \theta$  such that  $w(x) = e^{i\theta} \omega Q(\omega(x-x'))$ ).

### 4) Equation (II)

The problem of singularity for equation (II) has been studied in the last 20 years, and we give here part of results obtained.

i) No blow-up for small data

In [W1], it has been proved that for  $u \in H^1$ , we have the following

$$1/4 \int |u|^4 \leq 1/2 \int |\nabla u|^2 \{ \int |u|^2 / \int Q^2 \}.$$

It follows from this identity that if

$$\|u_0\|_{L^2} < \|Q\|_{L^2}$$

then there is non blow-up phenomon and the solution is globally defined in time.

ii) blow-up for large data

For this equation there are two way to obtain blow-up solutions.

- explicit blow-up solution.

From the conformal invariance of the equation if  $w(x)$  is a real solution of the equation (III),

then

$$1/t w(x/t) \exp(-i/t + ilx^2/4t)$$

is also a solution of equation (II) which blow-up at  $T = 0$ .

In particular  $S(t,x) = 1/t Q(x/t) \exp(-i/t + ilx^2/4t)$  is a blow-up solution such that  $\|u_0\|_{L^2} = \|Q\|_{L^2}$ .

-Viriel identity.

From [SoSyZ], [Gla], we have the following property of the solution of equation (II).

Assume that  $\|x\|u_0 \in L^2$  then for all time  $t$ ,  $\|x\|u_0 \in L^2$  and

$$d^2t/dt^2 \left\{ \int |x|^2 |u(t,x)|^2 dx \right\} = 16 E(u_0).$$

From this viriel identity, we have that

if  $E(u_0) < 0$  then the solution blow-up in finite time ( $T < +\infty$ ).

iii) Minimal blow-up solutions

Since if  $\|u_0\|_{L^2} < \|Q\|_{L^2}$  then there is non blow-up, and there is blow-up solution in the case where  $\|u_0\|_{L^2} = \|Q\|_{L^2}$ , one can ask is it possible to characterize all minimal blow-up solutions in  $L^2$  (that is solution which blows-up and such that  $\|u_0\|_{L^2} = \|Q\|_{L^2}$ ).

In [M1] (see also [M4] for a another approach of the proof), the following is proved.

Assume that  $u(t)$  is a blow-up solution with minimal mass (and  $u(t)$  is an  $H^1$  solution of equation (II)), that is  $\|u_0\|_{L^2} = \|Q\|_{L^2}$ . Then up to the invariance of the equation, we have

$$u(t,x) = S(t,x) = 1/t Q(x/t) \exp(-i/t + ilx^2/4t)$$

(that is there exist  $x', x'', \omega, \theta$  such that  $u(t,x) = e^{i\theta} \omega/t Q((x-x')/\omega/t - x'') \exp(-i\omega^2/t + ilx-x'|^2/4t)$ ).

## 5) Equation (I)

Until recently, there were no results on existence of solutions which blow-up for Zakharov equations. Indeed the two ingredients; the conformal invariance and the viriel identity which give blow-up results for the limit equation as  $c_0$  goes to infinity do not hold. We can note that there were numerical evidence of singular behavior of solution of equation (I) in [LPSSW] and [PSSW].

i) No blow-up for small data

One can show (see [AA1],[SS]) as for the Schrödinger equation, that if  $\|u_0\|_{L^2} < \|Q\|_{L^2}$  then there is non blow-up phenomenon and the solution is globally defined in time.

ii) blow-up for large data

As for equation (II), we are able to construct explicit blow-up solution and give obstructions to regular behavior.

- explicit blow-up solutions.

We do not have anymore the conformal invariance to obtain explicit blow-up solutions. We use in fact a bifurcation argument at "infinity" (using the structure of the nonlinear Schrödinger equation) to obtain explicit blow-up solution.

In [GIM1], a family of blow-up solutions in the energy space of the form

$$u(t,x) = \omega/t P(\omega x/t) \exp(-\omega^2 i/t + ilx^2/4t)$$

$$n(t,x) = \{\omega/t\}^2 N(\omega x/t)$$

where  $P(x) = P(|x|)$  and  $N(x) = N(|x|)$   
and

$$P = \Delta P + NP$$

$$(c_0\omega)^{-2} \{ r^2 N_{rr} + 6r N_r + 6N \} - \Delta N = \Delta P^2$$

is investigated.

More precisely, it is proved using this kind of construction, that there are blow-up solutions such that  $\|u_0\|_{L^2} = \|Q\|_{L^2} + \varepsilon$ , for all  $\varepsilon > 0$ .

We can note that the solutions constructed are numerically stable (see [LPSSW]). The problem now is the following, we have construct blow-up solutions but we do not existence of many (or a large set) of singular solution. For this purpose, we use a different approach.

-viriel identity.

In [M2], it is derived a perturbed viriel identity for the Zakharov equation. More precisly, for a regular solution with decay at infinity we have

$$d^2t/dt^2 \{ 1/4 \int |x|^2 |u(t,x)|^2 dx + c_0^{-2} \int_0^t \int (x \cdot v(t,x)) n(t,x) dx dt \} = 2H(u_0, n_0, v_0) - c_0^{-2} \int |v(t,x)|^2 dx.$$

From this pertubed viriel identity, we have in [M2] that

if  $H(u_0, n_0, v_0) < 0$  and the initial data are radially symmetric then the

solution blow-up in finite time ( $T < +\infty$ ) or in infinite time in  $H^1$  (with a concentration of  $u(t)$  in  $L^2$  as  $t$  goes to infinity).

We suspect that in the case where  $H(u_0, n_0, v_0) < 0$  then the solution away blows-up in finite time. This result give in particular the existence of a large class of singular solutions.

iii) Minimal blow-up solutions

Since if  $\|u_0\|_{L^2} < \|Q\|_{L^2}$  then there is non blow-up, and there are blow-up solution such that in the case where  $\|u_0\|_{L^2} = \|Q\|_{L^2} + \varepsilon$ , for all  $\varepsilon > 0$ . one can ask, as for the nonlinear Schrödinger equation about minimal blow-up solutions (that is solution which blows-up and such that  $\|u_0\|_{L^2} = \|Q\|_{L^2}$ ).

In [GIM2], we in fact proved that there is no minimal blow-up solution:

if  $\|u_0\|_{L^2} = \|Q\|_{L^2}$  then there is non blow-up.

Therefore, the situation is different from the one of the nonlinear Schrödinger equation.

iv) Instability and stability results of blow-up behavior

Let us first recall some results for the nonlinear Schrödinger equation. We have explicit blow-up solution such that the blow-up rate in  $H^1$  is of the type  $1/(T-t)$ . In particular, the one which the minimal blow-up solution has this rate of blow-up. From a physical point of view, we can expect that this rate is stable. It is not the case. Indeed, in [LPSS] for example it is observed numerically blow-up rate of the type  $\text{Log}|\text{Log}|T-t|| / (T-t)^{1/2}$ .

In [M3], it show for the Zakharov equation (with  $c_0$  finite but eventually very large), that the blow-up rate is stonger than  $1/(T-t)$ . More precisly, let  $(u, n, v)(t)$  a blow-up solution and  $T$  its blow-up time, we have for  $t$  near

$$\|u(t)\|_{H^1} \geq c/(T-t).$$

This shows that in fact the blow-up rate of the type  $\text{Log}|\text{Log}|T-t|| / (T-t)^{1/2}$  is unstable with respect to perturbations of the equation (with a term involving a wave equation). In contrary, the one with blow-up rate  $1/(T-t)$  seem numerically stable. This in particular shows the physical interest of the minimal blow-up solution of the nonlinear Schrödinger equation : the solution of the form

$$S(t,x) = 1/t Q(x/t) \exp(-i/t + i|x|^2/4t).$$

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