On complex-valued solutions to a 2D eikonal equation

Giorgio Talenti
Università di Firenze, Dipartimento di Matematica,
viale Morgagni 67A, 50134 Firenze, Italy.
e-mail address: talenti@udini.math.unifi.it

1 Introduction. Let n be a *nonnegative* sufficiently smooth function of two real variables x and y, and let *complex-valued* solutions w to the following first-order partial differential equation

$$w_x^2 + w_y^2 + n^2(x,y) = 0 \tag{1.1}$$

be in demand. Equation (1.1) arises, e.g., in questions about characteristic surfaces of Laplace's equation, and in the theory of diffraction by J.Keller and D.Ludwig. In the present paper we outline forthcoming results by R.Magnanini and the author, which include an existence theorem and a theorem about critical points.

The present section is devoted to formal remarks. Suppose

$$w = u + iv\,; \tag{1.2a}$$

i.e., suppose

$$u = \Re(w) \quad \text{and} \quad v = \Im(w)\,, \tag{1.2b}$$

the real and imaginary parts of w. Equation (1.1) is equivalent to the following first-order system of partial differential equations

$$u_x^2 + u_y^2 + n^2 = v_x^2 + v_y^2\,, \tag{1.3a}$$

$$u_x v_x + u_y v_y = 0 \tag{1.3b}$$

— in alternative notations,

$$|\nabla u|^2 + n^2 = |\nabla v|^2\,,$$
$$\nabla u \cdot \nabla v = 0\,.$$

Equation (1.3a) implies that the length of the gradient of v exceeds n, and equals n exactly at the critical points of u. Equation (1.3b) tells us that the gradients of u and v are orthogonal — thus the level lines of u are lines of steepest descent of v and the lines of steepest descent of u are level lines of v.

System (1.3) can be easily decoupled. In fact, algebraic manipulations show that (1.3) can be recast either in the following form

$$\begin{bmatrix} v_x \\ v_y \end{bmatrix} = \pm\sqrt{1+\frac{n^2}{u_x^2+u_y^2}}\begin{bmatrix} -u_y \\ u_x \end{bmatrix}, \tag{1.4a}$$

or in the form of the following pair

$$\begin{bmatrix} u_x \\ u_y \end{bmatrix} = \pm\sqrt{1-\frac{n^2}{v_x^2+v_y^2}}\begin{bmatrix} v_y \\ -v_x \end{bmatrix}, \quad v_x^2+v_y^2 \geq n^2 . \tag{1.4b}$$

Loosely speaking, equations (1.4a) make the gradient of v available if and only if u obeys the following partial differential equation

$$\frac{\partial}{\partial x}\left\{\sqrt{1+\frac{n^2}{u_x^2+u_y^2}}\, u_x\right\} + \frac{\partial}{\partial y}\left\{\sqrt{1+\frac{n^2}{u_x^2+u_y^2}}\, u_y\right\} = 0 ; \tag{1.5a}$$

the equations appearing in (1.4b) make the gradient of u available if and only if v obeys the constraint involved and the following partial differential equation

$$\frac{\partial}{\partial x}\left\{\sqrt{1-\frac{n^2}{v_x^2+v_y^2}}\, v_x\right\} + \frac{\partial}{\partial y}\left\{\sqrt{1-\frac{n^2}{v_x^2+v_y^2}}\, v_y\right\} = 0 . \tag{1.6a}$$

Thus, system (1.3) is satisfied if and only if either u satisfies equation (1.5a) and v is given by (1.4a), or v satisfies $v_x^2 + v_y^2 \geq n^2$ and equation (1.6a) and u is given by (1.4b).

Equations (1.5a) and (1.6a) can be recast in the following form

$$\left\{ |\nabla u|^4 + n^2 u_y^2 \right\} u_{xx} - 2n^2 u_x u_y u_{xy} + \left\{ |\nabla u|^4 + n^2 u_x^2 \right\} u_{yy} + n\, |\nabla u|^2\, \nabla n \cdot \nabla u = 0 , \tag{1.5b}$$

and

$$\left\{ |\nabla v|^4 - n^2 v_y^2 \right\} v_{xx} + 2n^2 v_x v_y v_{xy} + \left\{ |\nabla v|^4 - n^2 v_x^2 \right\} v_{yy} - n\, |\nabla v|^2\, \nabla n \cdot \nabla v = 0 , \tag{1.6b}$$

respectively — i.e., in the form of semilinear second-order partial differential equations with polynomial nonlinearities. If the coefficients of u_{xx}, u_{xy}, u_{yy} appearing on the left-hand side of (1.5b) are denoted by $a, 2b, c$, then

$$\begin{vmatrix} a & b \\ b & c \end{vmatrix} = |\nabla u|^6 \left(|\nabla u|^2 + n^2 \right) \geq 0 .$$

Hence equation (1.5b) should be qualified *elliptic* or *elliptic-parabolic* — notice that degeneracies occur at the critical points of solutions. If $a, 2b, c$ denote the coefficients of v_{xx}, v_{xy}, v_{yy} appearing on the left-hand side of (1.6b), then

$$\begin{vmatrix} a & b \\ b & c \end{vmatrix} = |\nabla v|^6 \left(|\nabla v|^2 - n^2 \right).$$

Hence solutions v to (1.6b), such that $|\nabla v|^2 > n^2$, are elliptic; any real-valued solution v to the equation

$$|\nabla v|^2 = n^2$$

— the standard equation of geometrical optics, which implies equation (1.6c) indeed — is a parabolic solution to (1.6c).

Observe that a set of terms, appearing on the left-hand side of equations (1.5b) and (1.6b), has a special geometric meaning. In fact, equations (1.5b) and (1.6b) read

$$|\nabla u| \Delta u + n^2 h + n \nabla n \cdot \frac{\nabla u}{|\nabla u|} = 0 , \tag{1.5c}$$

and

$$|\nabla v| \Delta v - n^2 k - n \nabla n \cdot \frac{\nabla v}{|\nabla v|} = 0 , \tag{1.6c}$$

respectively. Here

$$h = div \frac{\nabla u}{|\nabla u|} = \left(u_x^2 + u_y^2\right)^{-3/2} \left(u_y^2 u_{xx} - 2 u_y u_x u_{xy} + u_x^2 u_{yy}\right), \tag{1.7a}$$

$$k = div \frac{\nabla v}{|\nabla v|} . \tag{1.8a}$$

Recall from differential geometry that the absolute value of h at a point (x,y), where the gradient of u does not vanish, is the *curvature* at (x,y) of the *level line* of u crossing (x,y); the absolute value of k at a point (x,y) is the *curvature* at (x,y) of the *level line* of v crossing (x,y). Equations (1.4) yield

$$\pm h = div \frac{1}{|\nabla v|} \begin{bmatrix} v_y \\ -v_x \end{bmatrix} = \left(v_x^2 + v_y^2\right)^{-3/2} \left\{ -v_x v_y \left(v_{xx} - v_{yy}\right) + \left(v_x^2 - v_y^2\right) v_{xy} \right\}, \tag{1.7b}$$

$$\pm k = div \frac{1}{|\nabla u|} \begin{bmatrix} -u_y \\ u_x \end{bmatrix} ; \tag{1.8b}$$

hence the absolute value of h at a point (x,y) is also the curvature at (x,y) of the line of *steepest descent* of v crossing (x,y), and the absolute value of k at a point (x,y) is also the curvature at (x,y) of the line of *steepest descent* of u crossing (x,y).

2 An existence theorem. The existence of solutions to equation (1.5b), that take prescribed boundary values, can be settled in the following way.

Theorem 1. Let G be an open subset of the euclidean plane, having finite area — the ground domain; let g be a real-valued function from Sobolev space $W^{1,2}(G)$ — the boundary datum. Suppose n is bounded and belongs to $W^{1,2}(G)$. Then a real-valued function u exists such that: (i) u is in $W^{1,2}(G)$, u has second-order generalized derivatives and satisfies

$$\int_{G(\delta)} \frac{|\nabla u|^2}{n^2 + |\nabla u|^2}\left(u_{xx}^2 + 2u_{xy}^2 + u_{yy}^2\right) dxdy \leq \tag{2.1}$$

$$\delta^{-2}\int_G \left(n^2 + |\nabla u|^2\right) dxdy + \int_G |\nabla n|^2\, dxdy$$

for every positive δ — here $G(\delta) = \{(x,y) \in G\colon \operatorname{dist}((x,y), \partial G) > \delta\}$; (ii) u satisfies equation (1.5b) *almost everywhere* in G, u is a *viscosity solution* to (1.5b); (iii) u fits g on ∂G, i.e., $u - g \in W_0^{1,2}(G)$.

Proof, outlined. Let

$$j(x,y;\rho) = \int_0^\rho \sqrt{n^2(x,y) + t^2}\, dt\,, \tag{2.2a}$$

and let a functional J_ε be defined by

$$J_\varepsilon(u) = \int_G j\left(x, y; \sqrt{\varepsilon^2 + |\nabla u|^2}\right) dxdy \tag{2.2b}$$

for every nonnegative ε. Formulas (2.2) guarantee that J_ε is *strictly convex* and *coercive*, i.e., satisfies

$$J_\varepsilon(u) \geq \tfrac{1}{2}\int_G |\nabla u|^2\, dxdy$$

for every u from $W^{1,2}(G)$. Consequently, the following variational problem

$$\begin{aligned} &J_\varepsilon(u) = \text{minimum} \\ &\text{under the condition: } u - g \in W_0^{1,2}(G) \end{aligned} \tag{2.3}$$

has a *unique solution* — call it u_ε.

Observe that equation (1.5a) is exactly the Euler equation of functional J_0. However, u_0 *need not* satisfy such an equation. In fact, the following expansion

$$j(x,y,\rho)=n(x,y)\,\rho\Big\{1+n^{-2}(x,y)\,\rho^2+\cdots\Big\},$$

which holds for sufficiently small ρ, shows that J_0 is *not* differentiable at any function having critical points.

The main ingredient of the proof are statements (i)-(iii) below.

(i) If $\varepsilon>0$, J_ε is smoothly differentiable; moreover, u_ε satisfy the relevant Euler equation both in the following weak form

$$\frac{\partial}{\partial x}\left\{\sqrt{1+\frac{n^2}{|\nabla u|^2+\varepsilon^2}}\,\frac{\partial u_\varepsilon}{\partial x}\right\}+\frac{\partial}{\partial y}\left\{\sqrt{1+\frac{n^2}{|\nabla u|^2+\varepsilon^2}}\,\frac{\partial u_\varepsilon}{\partial y}\right\}=0 \tag{2.4a}$$

and in the following stronger form

$$\varepsilon^2\Big(n^2+2\,|\nabla u|^2+\varepsilon^2\Big)\Delta u+ \tag{2.4b}$$

$$\left\{|\nabla u_\varepsilon|^4+n^2\Big(\frac{\partial u_\varepsilon}{\partial y}\Big)^2\right\}\frac{\partial^2 u_\varepsilon}{\partial x^2}-2n^2\,\frac{\partial u_\varepsilon}{\partial y}\,\frac{\partial u_\varepsilon}{\partial x}\,\frac{\partial^2 u_\varepsilon}{\partial x\,\partial y}+\left\{|\nabla u_\varepsilon|^4+n^2\Big(\frac{\partial u_\varepsilon}{\partial x}\Big)^2\right\}\frac{\partial^2 u_\varepsilon}{\partial y^2}+$$

$$n\Big(\,|\nabla u|^2+\varepsilon^2\Big)\nabla n\cdot\nabla u=0$$

— in particular, u_ε has locally square-integrable second-order partial derivatives.

(ii) If $\varepsilon>0$, u_ε satisfies the following inequality

$$\int_{G(\delta)}\frac{|\nabla u_\varepsilon|^2}{n^2+|\nabla u_\varepsilon|^2}\left\{\Big(\frac{\partial^2 u_\varepsilon}{\partial x^2}\Big)^2+2\Big(\frac{\partial^2 u_\varepsilon}{\partial x\,\partial y}\Big)^2+\Big(\frac{\partial^2 u_\varepsilon}{\partial y^2}\Big)^2\right\}dxdy\le \tag{2.5}$$

$$\delta^{-2}\int_G\Big(n^2+|\nabla u_\varepsilon|^2\Big)\,dxdy+\int_G|\nabla n|^2\,dxdy$$

for every positive δ — observe that the constants involved in inequality (2.5) are *independent of* ε.

(iii) J_ε converges *uniformly* to J_0 as $\varepsilon\downarrow 0$, more precisely

$$0\le J_\varepsilon(u)-J_0(u)\le\int_G j(x,y;\varepsilon)\,dxdy \tag{2.6}$$

for every u in $W^{1,2}(G)$.

Statements (i)-(iii) allows one to infer that u_ε converge to u — in a topology stronger than the topology of $W^{1,2}(G)$ and the topology of any $W^{1,p}_{loc}(G)$ — as ε goes to zero, and that u_0 is the sought solution u. □

3 Critical points. As a rule, the critical points of solutions to second-order 2D partial differential equations are isolated. The following theorem shows that equation (1.5b) has the opposite property.

Theorem 2. Suppose

$$n(x,y) \geq \text{Constant} \; > 0 \,. \tag{3.1}$$

Let w be a smooth solution to equation (1.1), and let u be the real part of w. Then u *cannot* have isolated critical points.

Proof. Let

$$\boldsymbol{l} = \frac{1}{\mid \nabla u \mid} \begin{bmatrix} -u_y \\ u_x \end{bmatrix}, \tag{3.2a}$$

a unit vector field whose trajectories are the level lines of u and whose divergence is — in absolute value — the curvature of the lines of steepest descent of u. Equation (3.2a) gives

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \boldsymbol{l} = \frac{\nabla u}{\mid \nabla u \mid}, \tag{3.2b}$$

a unit vector field whose trajectories are the lines of steepest descent of u and whose divergence is — in absolute value — the curvature of the level lines of u.

Crucially, equations (1.4) and (3.2), and hypothesis (3.1) imply that $\boldsymbol{l}$ is *smooth everywhere*, even across the critical points of u.

As a consequence, the level lines of u are free from singular points, and the lines of steepest descent have a smooth curvature. Since

$$\nabla \mid \nabla u \mid \; = \begin{bmatrix} u_{xx} & u_{xy} \\ u_{xy} & u_{yy} \end{bmatrix} \frac{\nabla u}{\mid \nabla u \mid}, \tag{3.3}$$

we infer also that $\mid \nabla u \mid$ is continuously differentiable everywhere, even near the critical points of u.

An inspection shows that

$$\frac{\partial}{\partial \boldsymbol{l}} \mid \nabla u \mid \; + (div\, \boldsymbol{l}) \mid \nabla u \mid \; = 0 \,. \tag{3.4}$$

In conclusion, if u has a critical point — the origin, say — then the critical points of u must spread along the level line of u which crosses the origin. □