LOCAL WELL POSEDNESS OF NONLINEAR SCHRÖDINGER EQUATIONS

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In this note I will describe some recent work, done jointly with Gustav Ponce and Luis Vega, on non-linear Schrödinger equations.

We consider equations of the form

$$(N.L.S) \qquad \begin{cases} i\dfrac{\partial u}{\partial t} + \Delta u + F(u, \bar{u}, \nabla_x u, \nabla_x \bar{u}) = 0 \\ u(x, 0) = u_0(x) \ , \\ \text{where } x \in \mathbf{R}^n, t \in [0, T] \end{cases}$$

Here $F : \mathbf{C}^{2n+2} \to \mathbf{C}$ is a polynomial having no constant or linear terms. We are interested in establishing local well posedness ($\ell.w.p.$) results, and global well posedness ($g.w.p.$) results, with data in Sobolev spaces. By well posedness in X, we will mean that if $u_0 \in X$, the solution u will belong to $C([0,T]; X) \cap L^\infty([0,T]; X)$, it will be unique in that class, and the mapping $u_0 \mapsto u$ is continuous.

The case when $F(u, \bar{u}, \nabla_x u, \nabla_x \bar{u}) = G(u, \bar{u})$ has been extensively studied. In this case the standard energy estimate applies, and one can obtain ($\ell.w.p.$) in $H^s(\mathbf{R}^n)$, $s > u/2$. When $G(u, \bar{u}) = f(|u|)u$, and f is real valued, many more refined results have been obtained, using mixed norm estimates (the so called "Strichartz estimates") and a contraction in suitable spaces. The general case has been treated via the energy method, provided that one can show

$$(1.1) \qquad \left| \sum_{|\alpha| \leq s} \int_{\mathbf{R}^n} \partial_x^\alpha F(u, \bar{u}, \nabla_x u, \nabla_x \bar{u}) . \partial_X^\alpha u dx \right| \leq C_s (1 + \|u\|_{H^s}^\rho \|u\|_{H^s}^2 \ ,$$

for any $u \in H^s(\mathbf{R}^n)$, with $s > \frac{n}{2} + 1$ and $\rho = \rho(F) \in \mathbf{Z}^+$.

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It is clear that the estimate (1.1) can only be guaranteed if F exhibits an appropriate symmetry. For example :

$n=1\ ,\ F=\partial x(|u|^k.u)\ ,\ k\in \mathbf{Z}^+$

$n\geq 1\ ,\ F=F(u,\bar{u},\nabla_x\bar{u})$

$n\geq 1$ and $D_{\partial_{x_j}u}F\ ,\ D_{\partial_{x_j}\bar{u}}F\ ,\ j=1,\ ,n$ are real valued functions. Results of this type have been obtained by M. Tsutsumi and I. Fukuda (1980,1981), S. Klainerman (1988), S. Klainerman and G. Ponce (1983), J. Shatah (1982) and others. The proof follows the argument used for quasilinear symmetric hyperbolic systems. For those F's the same proof works if one removes the term involving the Laplacian in $(N.L.S.)$. In other words, this local result does not use the dispersive structure of the equations.

The results that I'm going to describe now apply to a general F in $(N.L.S.)$, even though (1.1) may fail, by exploiting in a crucial way the dispersive nature of the equation. The first progress in this direction was made by Kenig-Ponce-Vega in 1991, who obtained :

Theorem A.— *There exists s_0, $m\in\mathbf{Z}^+$, depending only on n, and $\delta>0$, depending only on F and n such that, for $u_0\in H^s(\mathbf{R}^n)\cap L^2(\mathbf{R}^n,|x|^m dx)=X$, and, $s\geq s_0$,*

$$\|u_0\|_{H^{s_0}(\mathbf{R}^n)}+\|u_0\|_{L^2(\mathbf{R}^n,|x|^m dx)}\leq\delta \tag{1.2}$$

$(N.L.S.)$ is $(\ell.w.p.)$ in X. Moreover, if $\partial^2F(0)=0$, we can take $m=0$.

The same method also yielded global well-posedness.

Theorem B.— *(Kenig-Ponce-Vega 1994) If in addition $\partial^\alpha F(0)=0\ ,\ |\alpha|\leq 4$, then we can take $m=0$, and under (1.2), global well posedness holds.*

The proofs were obtained through Picard iteration in suitable function spaces, using a "local smoothing effect" for the free Schrödinger equation, to exploit the dispersion. Of course, a drawback of these results is the smallness assumption (1.2) used to obtain a local (in time) well posedness result.

Let us briefly describe this "local smoothing effect", and the reason for (1.2) in Theorem A.

Let $e^{it\Delta}u_0=\int e^{i(x\xi+t|\xi|^2)}\hat{u}_0(\xi)d\xi$ denote the solution to the free Schrödinger equation. Clearly, $e^{it\Delta}$ is a unitary group on $L^2(\mathbf{R}^n)$, and hence, we have

$$\sup_{0<H<T}\|e^{it\Delta}u_0\|_{H^s}\leq C\|u_0\|_{H^s}\ . \tag{1.3}$$

The "local smoothing effect" can be described in the following manner : let us write $\mathbf{R}^n=\cup_j Q_j$, where the Q_j are unit size, non-overlapping cubes. For a function w on $\mathbf{R}^n\times[0,T]$, we let

$$|||w|||_T=\sup_j\|w\|_{L^2(Q_j\times[0,T])} \tag{1.4}$$

$$|||w|||'_T = \Sigma_j \|w\|_{L^2(Q_j\times[0,T])}$$

The "local smoothing effect" is then

$$|||D_x^{1/2} e^{it\Delta} u_0|||_T \le C\|u_0\|_{L^2} \ . \tag{1.5}$$

This was proved independently by Constantin-Saut (1989), Vega (1988) and Sjölin (1987).

However, for the applications to $(N.L.S.)$ one needs to be able to absorb one full derivative ! Fortunately, one sees that $(N.L.S.)$ is equivalent, by Duhamel's formula, to

$$u(t) = e^{it\Delta}u_0 + \int_0^t e^{i(t-t')\Delta} F(u,\bar u,\nabla_x u,\nabla_x \bar u)dt' \ , \tag{1.6}$$

and thus, the gain of 1 derivative is only needed in the second term of the above formula.

Kenig, Ponce and Vega (1991), showed

$$|||D_x \int_0^t e^{i(t-t')\Delta} w dt'|||_T \le C|||w|||'_T \ , \tag{1.7}$$

and this was the main new ingredient in the proofs of Theorems A and B. To illustrate this application, assume, for example that $F = u^2\partial_x u$, and let's try to see if we can show that the right hand side of (1.6) is a contraction. Thus, let's estimate

$$|||D_x^{1/2+s}u|||_T \le C\|u_0\|_{H^s} + |||D^{1/2+s}(u^3)|||'_T + \quad \text{lower order terms,}$$

where we have used (1.5) and (1.7).

Now, we want to end, in the right hand side with the same norm we have on the left. Note that,

$$|||D_x^{1/2+s}(u^3)|||'_T = |||u^2 D_x^{1/2+s}(u)|||'_T + \quad \text{lower order terms}$$

$$\le \left(\Sigma_j \|u^2\|_{L^\infty(Q_j\times[0,T])}\right) |||D_x^{1/2+s}(u)|||_T + \quad \text{lower order terms.}$$

The L^∞ norm on the cubes $Q_j \times [0,T]$ is forced on us, and in order to have a contraction, the factor in front of $|||D_x^{1/2+s}(u)|||_T$ must be made small by taking T small. Since the norm is L^∞ in $Q_j \times [0,T]$, this forces us to have small data, and thus, (1.2).

In 1992, N. Hayashi and T. Ozawa were able to go back to the energy estimate, and see that, after a transformation of the equation, it does apply to give $(\ell.w.p.)$ without the smallness assumption (1.2). Their idea was to eliminate the bad first order term $\frac{\partial u}{\partial x}$, introducing the new function $h(X,t) =$

$$= u(x,t).\exp\left(-\frac{1}{2}\int_{-\infty}^x \mathrm{Im}\frac{\partial F}{\partial_{\frac{\partial u}{\partial x}}}(y,t)dy\right) \ .$$

The disadvantage of this method is that it does not allow one to obtain the smoothing estimate $|||D_x^{1/2+s}u|||_T < \infty$, and it does not obtain the solution by Picard iteration. In 1994, H. Chiara succeeded in removing the smallness condition (1.2) for all dimensions n, by extending the idea of Hayashi and Ozawa to higher dimensions, by means of pseudodifferential operators, and then using a version of the energy method. I will now describe an alternative approach to this result, which combines the ideas used to prove Theorem A with Chihara's approach, and which leads to some further results, like establishing local smoothing estimates for solutions of $(N.L.S.)$ and also allows one to treat diagonal systems of the same type. Let us reconsider the $(N.L.S.)$ equation, with $F = u^2\partial_x u$, and data u_0, and let us rewrite in the form

$$(1.8) \qquad \begin{cases} i\dfrac{\partial u}{\partial t} + \Delta u + u_0^2(x)\partial_x u + (u^2 - u_0^2)\partial_x u = 0 \\ u(x,0) = u_0(x) \end{cases}$$

Let $W^{u_0}(t)$ be the "group" giving the solution to the linear problem

$$\begin{cases} i\dfrac{\partial w}{\partial t} + \Delta w + u_0^2(x)\dfrac{\partial w}{\partial x} = 0 \\ w(x,0) = w_0(x) \end{cases}$$

Then, the solution to $(N.L.S.)$ can be written as

$$u(t) = W^{u_0}(t)u_0 + \int_0^\gamma W^{u_0}(t-t')[(u^2 - u_0^2)\partial_x^u]\ .dt'$$

Thus, if we have the analog of the estimates (1.3), (1.5) and (1.7) for W^{u_0}, the "smallness" in L^∞ of $(u^2 - u_0^2)$ is guaranteed, for T small, and our previous approach works.

We are thus lead to studying the following linear problem :

$$(L.S.) \qquad \begin{cases} i\dfrac{\partial u}{\partial t} + \Delta u + \vec{b}(x).\nabla u + \vec{\beta}(x).\nabla \bar{u} + C(x)u + \gamma(x)\bar{u} = 0 \\ u(x,0) = u_0 \end{cases}$$

And, we need to establish, for solutions of $(L.S.)$ estimates of the type :

$$(1.9) \qquad \begin{cases} (i) \quad \sup\limits_{|t|\le T} \|u(.,t)\|_{H^s} \le C_{s,T}\|u_0\|_{H^s} \\ (ii) \quad |||D_x^{1/2+s}u|||_T \le C_{s,T}\|u_0\|_{H^s} \\ (iii) \quad |||D_x \displaystyle\int_0^t W(t-t')v\ dt'|||_T \le C_T|||v|||'_T\ , \end{cases}$$

where $W(t-t')$ denotes the solution operator to $(L.S.)$ at time $t-t'$. We need that the constants in (1.9) depend on s, T and on appropriate (Sobolev) worms of $\vec{b}, \vec{\beta}, c, \vec{\gamma}$.

This type of problem, when $\vec{\beta} \equiv 0, \gamma \equiv 0$, has been studied for a long time. In fact, when also $C \equiv 0$, Mizohata (1985) proved that, a necessary condition for (1.9) (i) in the case $s = 0$ is

$$\left| Re \int_0^r \vec{b}(x + \omega.s).\omega ds \right| \leq C \tag{1.10}$$

for all $(x, r) \in \mathbf{R}^n \times \mathbf{R}$, $\omega \in S^{n-1}$. For instance, if $\vec{b} = \vec{e}_1$, the unit vector in the x_1 direction, (1.10) fails, and so does (1.9). (This can be easily checked directly). Moreover, Mizohata showed that, if $D^\alpha \vec{b}$ verifies (1.10), $|\alpha| \leq N(n)$, then (1.9) (i) holds, for $s = 0$. Note that in the example arising from (1.8), $u_0^2(X) \in L^1$, and thus, the decay assumption (1.10) holds. Our main result in this direction is

Theorem C.— *(Kenig-Ponce-Vega 1995) Under appropriate decay and differentiability assumptions on $\vec{b}, \vec{\beta}, \gamma, C$, (1.9) holds.*

Corollary D.— *Theorem A holds without the smallness assumption (1.2).*

I will now give a brief sketch of our proof of Theorem C. Full proofs of Theorem C, Corollary D, and further extensions and applications will appear elsewhere.

Step 1 : The estimates (1.9) hold for the following problem :

$$\begin{cases} i\dfrac{\partial w}{\partial t} + \Delta w + C(w) + \tilde{C}(\bar{w}) = 0 \\ w(X, 0) = w_0(x) \end{cases}$$

Here, C and $\widetilde{C}$ are classical pseudodifferential operators, in the x variable, of order 0. The proof of step 1 consists in observing that C, $\widetilde{C}$ are bounded in $L^\infty(H^s)$ by classical results and in $|||\ |||_T$, and $|||\ |||'_T$ by simple arguments. One then writes

$$w(t) = e^{it\Delta} w_0 + \int_0^t e^{i(t-t')\Delta}[C(w) + \widetilde{C}(\bar{w})]dt' \ ,$$

uses (1.3) (1.5) and (1.7) combined with this fact, which gives (1.9) for small enough T.

Step 2 : "Eliminate $\vec{\beta}$" In order to do so, we write, following H. Chihara, $(L.S.)$ as a system, and we diagonalize the system, in the manner used to treat symmetrizable hyperbolic systems. We let $\vec{w} = (u, \bar{u})$ and we rewrite $(L.S.)$ in terms of $\vec{w}$, as

$$\begin{cases} i\dfrac{\partial \vec{w}}{\partial t} + H\vec{w} + B\vec{w} + C\vec{w} = 0 \\ \vec{w}_{|t=0} = \vec{w}_0 \ , \end{cases}$$

where $H = \begin{pmatrix} \Delta & 0 \\ 0 & -\Delta \end{pmatrix}$, B is a 2×2 matrix of first order differential operators in x, C is a 2×2 matrix of functions, acting on $\vec{w}$ by multiplication. Our task is to diagonalize B. The symbol of B is of the form

$$\begin{pmatrix} b_{11}(x, \xi) & b_{12}(x, \xi) \\ b_{21}(x, \xi) & b_{22}(x, \xi) \end{pmatrix}$$

and we seek to make $b_{12}, b_{21} \equiv 0$. We apply on the right, to our system an invertible, matrix valued, 0 order pseudodifferential operator Λ, where $\Lambda = I - S$, and S is of order-1, with symbol $\begin{pmatrix} 0 & s_{12}(x,\xi) \\ s_{21}(x,\xi) & 0 \end{pmatrix}$, where s_{12}, s_{21} are to be formal. Errors of order 0 are acceptable, because of step 1.

$\Lambda H - H\Lambda =$ (modulo order 0) the operator whose symbol is

$$\begin{pmatrix} 0 & 2|\xi|^2 s_{12}(x,\xi) \\ -2|\xi|^2 s_{21}(x,\xi) & 0 \end{pmatrix}$$

$\Lambda B = B$ (modulo order 0), and we thus choose s_{12}, s_{21}, of order -1 so that

$$2|\xi|^2 s_{12}(x,\xi) = -b_{12}(x,\xi)$$

and $-2|\xi|^2 s_{21}(x,\xi) = -b_{21}(x,\xi)$. This "incouples" the system, and we have thus effectively eliminated $\vec{\beta}$.

Step 3 : (Eliminate $\vec{b}$) We now, effectively have $(L.S.)$ but with $\vec{\beta} \equiv 0$. We now seek to eliminate $\vec{b}$. We will apply to $(L.S.)$ a 0 order pseudodifferential operator Φ, elliptic, to be chosen. Terms of order 0 are still acceptable. Recall that $\Phi\Delta - \Delta\Phi =$ (modulo order 0) the operator whose symbol is $2i\Sigma_j \frac{\partial}{\partial x_j}\Phi(x,\xi)\xi_j$, and $\Phi b =$ (modulo order 0 the operator whose symbol is $i\Phi(x,\xi)\vec{b}(x).\xi$. Here b is the 1^{st} order differential operator $\vec{b}.\nabla$, and $\Phi(x,\xi)$ the symbol of Φ. We need to choose Φ, elliptic, of order 0, so that

$$2\Sigma_j \frac{\partial}{\partial x_j}\Phi(x,\xi)\xi_j = -\Phi(x,\xi)\vec{b}(x).\xi \ , \tag{1.11}$$

for large ξ.

Suppose that $\omega \in S^{n-1}$, we can explicitly solve

$$\omega.\nabla_x\Phi(x,\omega) = -\Phi(x,\omega)\frac{\vec{b}(x).\omega}{2}$$

by the formula

$$\Phi(x,\omega) = \exp(-\frac{1}{2}\int_0^{x.\omega} \vec{b}(x - x.\omega\omega + s\omega).\omega ds)$$

and then extend it for $|\xi| \geq 1$ to be homogeneous of degree 0. (1.10) and cooresponding conditions on the derivatives guarantee that the resulting symbol is in S^0. Note also that Φ is elliptic, which finishes the argument.