JOURNÉES ÉQUATIONS AUX DÉRIVÉES PARTIELLES

THOMAS KAPPELER

Smoothing of dispersive waves

Journées Équations aux dérivées partielles (1992), p. 1-4

<http://www.numdam.org/item?id=JEDP_1992____A10_0>

<u>Smoothing of dispersive waves</u>

Thomas KAPPELER
Dept of Mathematics
Ohio-State University
Colombus OH 43210 U.S.A.

This is a report on work in progress in collaboration with W. Crang and W. Strauss, concerning regularity of solutions of dispersive evolution equations.

I. We consider the following fully non linear evolution equations of KdV-type :

$$\partial_t u + f(\partial_x^3 u, \partial_x^2 u, \partial_x u, u) = 0 \quad x \in \mathbf{R}; t > 0 \qquad (1)$$

where f is a smooth real valued function satisfying the following assumptions :

(A1) $\frac{\partial f}{\partial(\partial_x^3 u)} \geq C > 0$

(A2) $\frac{\partial f}{\partial(\partial_x^2 u)} \leq 0$

The following theorem is a version of a more general result, proved in [CKS1] concerning the initial value problem

$$\begin{cases} \partial_t u + f = 0 & t > 0 \\ u/_{t=0} = \phi \end{cases}$$

<u>Theorem</u> : Assume $\phi \in H^7$. Then :

(1) There exists a maximal time of existence, $O < T^* \leq \infty$ for a unique solution $u \in L^\infty([0,T]; H^7)$ for all $T < T^*$.
(2) If in addition $\int_0^\infty |\partial_x^7 \phi|^2 x^M dx < \infty$ with $M \geq O$, then for all $T < T^*, 0 \leq k \leq M$ and $\delta > 0$

$$(i) \quad \sup_{0 \leq t \leq T} t^k \int_{-\infty}^{\infty} |\partial_x^{7+k} u(x,t)| w_{\delta, M-k}(x) dx < \infty$$

$$(ii) \quad \int_O^T t^M \int_{-\infty}^{\infty} |\partial_x^{7+M+1} u(x,t)|^2 w_{\delta,-1}(x) dx < \infty$$

where for $j \geq 0 \, w_{\delta,j}(x)$ are smooth increasing weight functions of the form :

$$w_{\delta,j}(x) = e^{\delta x}(x \leq 0) \quad w_{\delta,j}(x) = x^j (x \geq 1).$$

The weight function $w_{\delta,-1}(x)$ is supposed to be smooth, integrable in x and $w_{\delta,-1}(x) = e^{\delta x}(x \leq 0)$.

We remark that in the case $m = 0$ the result (ii) is usually referred to as a local gain of regularity which has been studied extensively in recent years (cf eg.[CS] and references mentionned in [CS], [CKS 1,2,3]).

2. We consider the following fully non linear evolution equations of Schrödinger type in one space dimension :

$$\frac{1}{i} \partial_t u + g(D_x^2 u, D_x u, u) = 0 \qquad x \in \mathbf{R}; t > 0 \qquad (2)$$

where u is now complex valued and g is a smooth, complex valued function with the properties :
(B1) $\frac{\partial g}{\partial (D_x^2 u)} \geq C > 0$
(B2) $\frac{\partial g}{\partial (D_x u)}$ is purely imaginary.

Assuming that the initial data ϕ is in H^5 , we show that there exists a maximal time of existence $T^*, 0 < T^* < \infty$, for a unique solution $u \in L^\infty([0,T]; H^5)$ for all $T < T^*$.

Gain of regularity results for special semilinear equations of type (2) have been obtained, among others, by Hayashi, Nakamitu and Tsutsumi ([HNT]). Generalizing their work, we obtain, under additionnal assumptions on g a gain of regularity result similar to part (2) of the theorem above for semilinear equations of type (2) (for details cf.[CKS2]).

3. Consider the following linear dispersive evolution equations in several space dimensions :

$$\begin{cases} i\partial_t u &= A(x,D)u \quad t>0; \ x\in\mathbf{R}^n & (3) \\ u(x,0) &= \phi(x) & (4) \end{cases}$$

where $u=u(x,t)$ is complex valued and $A(x,D)$ is a scalar valued differential generator in the $x-$variables of order m with smooth principal symbol $a(x,\xi)$.

The assumptions on $a(x,\xi)$ are the following :
(C1) $a(x,\xi)$ is dispersive ,i.e.
(i) $a(x,\xi)$ is real valued
(ii) $|grad_\xi a(x,\xi)|\to\infty$ for $|\xi|\to\infty$
(C2) There exists a smooth function $a_\infty(\xi)$ of ξ alone such that

$$|\partial_x^\alpha\partial_\xi^\beta(a(x,\xi)-a_\infty(\xi))|\le C_{\alpha\beta}\frac{(1+|\xi|)^{m-|\beta|}}{(1+|x|)^{\rho(|\alpha|)}}$$

where $\rho(k)\ge k$ and $\rho(0)>1,\rho(1)>2$.
(C3) The bicharacteristic flow, induced by the Hamiltonian $a(x,\xi)$ on the cotangent space $T^*\mathbf{R}^n$ has no trapped rays.

Let us remark that in the case where $a(x,\xi)$ does not depend on x the definition(C1) of dispersiveness is due to Constantin and Saut [CS]. Further we note that the condition (C1) insures that $a(x,\xi)$ is a phase function in the sense of Trèves [Tr]. For equations of the form (3) results concerning propagation of singularities due to Boutet de Monvel [Bo] and Lascar [La] do hold. Similar to our results concerning equations (1) and (2) mentioned above, our interest is to go further and deduce from decay assumptions on the initial data regularity results for the solutions to the initial value problem (3) ,(4) in $t > 0$. Under natural assumptions on the lower order coefficients of $A(x, D)$ we prove a smoothing result one consequence of which is the following.

Theorem Assume $(C1) - (C3)$. If the initial data ϕ is in L^2 and compactly supported, then the unique solution $u(x, t)$ of (3),(4) is C^∞ for $t \neq 0$

References

[B0] L. Boutet de Monvel, "Propagation des singularités d'équations analogues à l'équation de Schrödinger ", Congrès Nice, 1974.

[CS] P. Constantin, J.C. Saut, "Local smoothing properties of dispersive equations", J. AMS 1(1988), 413-439.

[CKS 1] W. Crang, T. Kappeler, W. Strauss "Gain of regularity for equations of KdV type", Ann IHP, Analyse nonlinéaire 9(1992), 147-186.

[CKS 2] W. Crang, T. Kappeler, W. Strauss "Gain of regularity for nonlinear equations of Schrödinger type in one space dimension", in preparation.

[CKS 3] W. Crang, T. Kappeler, W. Strauss "Smoothing of linear dispersive waves", in preparation.

[HNT] N. Hayashi, K. Nakamitsu, M. Tsutsumi, "On solutions of the initial value problem for the nonlinear Schrödinger equations in one space dimension", Math.$\mathcal{Z}$, 192(1986),637-650.

[La] R. Lascar, "Propagation des singularités des solutions d'équations pseudodifférentielles quasi homogènes", Ann. Inst. Fourier 27(1977),79-123.

[Tr] F.Trèves, "An introduction to pseudodifferential operators and Fourier integral operators", Recife Univ. Fed. de Pernambuco, Inst. di Matematica, 1973.