GLOBAL ANALYTIC AND GEVREY SURJECTIVITY OF THE MIZOHATA OPERATOR $D_2 + ix_2^{2k}D_1$ (*)

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Global analytic surjectivity of all linear constant coefficient partial differential operators on $\mathbf{R}^2$ has been proved by E. De Giorgi and L. Cattabriga [5], where counterexamples for the case of operators on $\mathbf{R}^3$ are also indicated (see also E. De Giorgi [4] and for other results for operators on open sets of $\mathbf{R}^n$: T.Kawai [10] and L.Hörmander [8]).Global Gevrey surjectivity of all linear constant coefficient partial differential operators on $\mathbf{R}^2$ was subsequentely proved by L. Cattabriga [3] in the case of Gevrey spaces with rational index (see also L. Cattabriga [2], G. Zampieri [15], R.W. Braun-R. Meise-D. Vogt [1] for the case of operators on $\mathbf{R}^n$).As for the case of linear operators with variable coefficients results on global Gevrey surjectivity are given by L.Ehrenpreis [6] and in the analytic case by T.Kawai [11].

Here we consider,as a simple example of an operator on $\mathbf{R}^2$ with variable coefficients,the Mizohata operator $D_2 + ix_2^{2k}D_1$,where k is a positive integer and $D_j=-i\partial_j$, $j = 1,2$, and prove the following

Theorem. Let $\mathcal{E}^{\{s\}}(\mathbf{R}^n)$, $s \geq 1$, be the space of all C^∞ functions f on $\mathbf{R}^n$ such that for every compact subset K of $\mathbf{R}^n$ there exists a constant $A > 0$ such that

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$$\sup_{x\in K} \sup_{\alpha\in Z_+^n} A^{-|\alpha|} \alpha!^{-s} |\partial_x^\alpha f(x)| < +\infty.$$

Then for every $s \geq 1$

i) $(D_2 + ix_2^{2k}D_1)\ \mathcal{E}^{\{s\}}(R^2) = \mathcal{E}^{\{s\}}(R^2)$,

ii) $(D_2 + ix_2^{2k}D_1)\ \mathcal{E}^{\{s\}}(R^3) \subsetneq \mathcal{E}^{\{s\}}(R^3)$.

Proof. Let $f \in \mathcal{E}^{\{s\}}(R^2)$, $s \geq 1$, and let $\sigma > \max\{s,1\}$. Since $f \in \mathcal{E}^{\{\sigma\}}(R^2)$, from a result by H.Komatsu [12] it follows that there exists $v_1 \in \mathcal{E}^{\{\sigma\}}(R^2)$ such that

$$(1)\qquad \begin{cases} v_1(x_1,0) = 0 \\ \partial_x^\gamma\left[(D_2 + ix_2^{2k} D_1)v_1 - f\right](x_1,0) = 0 \qquad \forall\gamma\in Z_+^2,\ x_1\in R. \end{cases}$$

Let

$$(2)\qquad h(x_1, x_2) = f(x_1, x_2) - (D_2 + ix_2^{2k}D_1)v_1(x_1, x_2)\ ,$$

and with the new variables in R^2

$$\begin{cases} y_1 = x_1 \\ y_2 = x_2^{2k+1}/(2k+1), \end{cases}$$

set

$$g(y_1, y_2) = \begin{cases} h(y_1,[(2k+1)y_2]^{1/(2k+1)})[(2k+1)y_2]^{-2k/(2k+1)} & \text{for } y_2\neq 0 \\ 0 & \text{for } y_2=0. \end{cases}$$

In view of (1), $g\in C^\infty(R^2)$. Hence, by the global $C^\infty(R^2)$-surjectivity of all linear partial differential operators with constant coefficients, there exists $w\in C^\infty(R^2)$ such that

$$(D_2 + i D_1)w(y_1, y_2) = g(y_1, y_2) \qquad \text{on } R^2.$$

Letting

$$v_2(x_1, x_2) = w(x_1,x_2^{2k+1}/(2k+1)) \quad , \quad (x_1,x_2) \in R^2,$$

it follows that

$$v_2 \in C^\infty(R^2)$$

and

(3) $$(D_2 + ix_2^{2k}D_1)v_2(x_1,x_2) = x_2^{2k}g(x_1, x_2^{2k+1}/(2k+1)) = h(x_1,x_2), \ (x_1,x_2) \in \mathbf{R}^2.$$

Thus from (2) and (3)

$$(D_2 + ix_2^{2k}D_1)(v_1+v_2) = f \qquad \text{on } \mathbf{R}^2,$$

where

$$v = v_1+v_2 \in C^\infty(\mathbf{R}^2).$$

Recalling now that the Mizohata operator $(D_2+ix_2^{2k}D_1)$ is (analytic and) s-Gevrey hypoelliptic on $\mathbf{R}^2$ for every $s \geq 1$ (see for example S.Mizohata [13], L.Rodino [14], and also L.Hörmander [9]), we conclude that v is in fact in the same space $\mathcal{E}^{\{s\}}(\mathbf{R}^2)$ as f is. This proves part i) of the theorem.

To prove part ii), let $f \in \mathcal{E}^{\{s\}}(\mathbf{R}^3)$, $s \geq 1$, and define

$$g(x_1,x_2,x_3) = f(x_1, x_2^{2k+1}/(2k+1), x_3)\ x_2^{2k} \qquad , \quad (x_1,x_2,x_3) \in \mathbf{R}^3.$$

Since $g \in \mathcal{E}^{\{s\}}(\mathbf{R}^3)$, if $(D_2 + ix_2^{2k}D_1)\ \mathcal{E}^{\{s\}}(\mathbf{R}^3) = \mathcal{E}^{\{s\}}(\mathbf{R}^3)$, there exists $w \in \mathcal{E}^{\{s\}}(\mathbf{R}^3)$ such that $(D_2+ ix_2^{2k}D_1)w(x_1,x_2,x_3) = g(x_1,x_2,x_3)$ on $\mathbf{R}^3$.

Hence $v(x_1,x_2,x_3) = w(x_1,[(2k+1)x_2]^{1/(2k+1)}, x_3)$ would be a $\mathcal{D}'(\mathbf{R}^3)$ solution of $(D_2+iD_1)v = f$. Since the operator D_2+iD_1 is partially hypoelliptic in $\mathbf{R}^3$ with respect to the (x_1,x_2) variables, from the corollary of Theorem 4.1.2 of [7], it follows that $v \in \mathcal{E}^{\{s\}}(\mathbf{R}^3)$ .So the operator $D_2 +iD_1$ on R^3 would be s-surjective,what is false as it has been proven in [3] and [4].

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