

ON THE SPECTRUM OF SINGULARLY PERTURBED OPERATORS

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Many problems of mathematical physics lead to the study of the behaviour of eigenvalues and eigenvectors of operators singularly depending on a parameter ε . In many cases these operators act in spaces which also depend on the parameter ε .

An abstract theorem is given here on the behaviour of eigenvalues and eigenvectors of a sequence of operators acting in spaces depending on a parameter ε as $\varepsilon \rightarrow 0$. This theorem can be applied to study asymptotic properties of eigenvalues and eigenfunctions of boundary value problems for elliptic equations and systems with rapidly oscillating coefficients in perforated domains with a periodic structure, to study spectral properties of a G -convergent sequence of operators, the spectrum of eigenvalue problems in domains with an oscillating boundary and many other problems. We give here the application of this abstract theorem to the Dirichlet problem for elliptic second order equations in a perforated domain and to the problem of the oscillation of systems with concentrated masses.

Let $\mathcal{K}_\varepsilon, \mathcal{K}_0$ be separable Hilbert spaces with the scalar products $(u^\varepsilon, v^\varepsilon)_\varepsilon, (u, v)_0$ respectively and $A_\varepsilon : \mathcal{K}_\varepsilon \rightarrow \mathcal{K}_\varepsilon, A_0 : \mathcal{K}_0 \rightarrow \mathcal{K}_0$ be linear continuous operators, $Im A_0 \subset \mathcal{V} \subset \mathcal{K}_0$, where \mathcal{V} is a linear subspace of \mathcal{K}_0 , $Im A_0 \equiv \{v : v = A_0 u, u \in \mathcal{K}_0\}, 0 < \varepsilon < 1$.

We assume that spaces $\mathcal{K}_\varepsilon, \mathcal{K}_0, \mathcal{V}$ and operators A_ε, A_0 satisfy the following conditions $C_1 - C_4$.

C_1 - There exist linear continuous operators $R_\varepsilon : \mathcal{K}_0 \rightarrow \mathcal{K}_\varepsilon$ such that for any $f^0 \in \mathcal{V}$:

$$(R_\varepsilon f^0, R_\varepsilon f^0)_\varepsilon \rightarrow \gamma (f^0, f^0)$$

as $\varepsilon \rightarrow 0$, where $\gamma = \text{const} > 0$ and γ does not depend on f^0 .

C_2 - Operators A_ε, A_0 are positive, compact and selfadjoint, moreover norms $\|A_\varepsilon\|_{\mathcal{L}(\mathcal{K}_\varepsilon)}$ are bounded by a constant which does not depend on ε .

C_3 - For any $f^0 \in \mathcal{V}$

$$\|A_\varepsilon R_\varepsilon f^0 - R_\varepsilon A_0 f^0\|_\varepsilon \rightarrow 0$$

as $\varepsilon \rightarrow 0$ where $\|u\|_\varepsilon \equiv (u, u)_\varepsilon^{1/2}$.

C_4 - The family of operators A_ε is uniformly compact in the following sense. From any sequence $f^\varepsilon \in \mathcal{X}_\varepsilon$, such that $\sup_\varepsilon \|f^\varepsilon\|_\varepsilon < \infty$, one can select a subsequence $f^{\varepsilon'}$ and a vector $w^0 \in \mathcal{X}^0$ for which

$$\|A_{\varepsilon'} f^{\varepsilon'} - R_{\varepsilon'} w^0\|_{\varepsilon'} \rightarrow 0 \quad \text{as } \varepsilon' \rightarrow 0$$

Let us consider the spectral problems :

$$(1) \quad A_\varepsilon u_\varepsilon^k = \mu_\varepsilon^k u_\varepsilon^k, \quad k = 1, 2, \dots, \quad u_\varepsilon^k \in \mathcal{X}_\varepsilon, \quad (u_\varepsilon^\ell, u_\varepsilon^m)_\varepsilon = \delta_{\ell m}$$

$$(2) \quad A_0 u^k = \mu_0^k u^k, \quad k = 1, 2, \dots, \quad u^k \in \mathcal{X}_0, \quad (u^\ell, u^m)_0 = \delta_{\ell m}$$

where $\delta_{\ell m}$ is Kronecker's symbol, eigenvalues $\mu_\varepsilon^k, \mu_0^k$ are numbered in nonincreasing order and every eigenvalue is counted as many times as its multiplicity.

The following theorem gives estimates for the difference of eigenvalues μ_ε^k and μ_0^k .

THEOREM 1: Let for spaces $\mathcal{X}_\varepsilon, \mathcal{X}_0$ and operators A_ε, A_0 conditions $C_1 - C_4$ be fulfilled. Then there exists a sequence $\beta_\varepsilon^k \rightarrow 0$ as $\varepsilon \rightarrow 0$, $0 < \beta_\varepsilon^k < \mu_0^k$, such that:

$$(3) \quad |\mu_\varepsilon^k - \mu_0^k| \leq \mu_0^k \gamma^{-\frac{1}{2}} (\mu_0^k - \beta_\varepsilon^k)^{-1} \sup_u \|A_\varepsilon R_\varepsilon u - R_\varepsilon A_0 u\|_\varepsilon$$

$k = 1, 2, \dots$, where the upper bound is taken over all u such that

$$u \in N(\mu_0^k, A_0) \equiv \{u \in \mathcal{X}_0, A_0 u = \mu_0^k u\}, \quad \|u\|_0 = 1.$$

Let $k \geq 0$, $m \geq 1$ be integers and $\mu_0^k > \mu_0^{k+1} = \dots = \mu_0^{k+m} > \mu_0^{k+m+1}$, i.e. the multiplicity of the eigenvalue μ_0^{k+1} of problem (2) is equal to m . Then for any $w \in N(\mu_0^{k+1}, A_0)$, $\|w\|_0 = 1$, there exists a linear combination u^ε of eigenvectors $u_\varepsilon^{k+1}, \dots, u_\varepsilon^{k+m}$ of problem (1) such that

$$(4) \quad \|u^\varepsilon - R_\varepsilon w\|_\varepsilon \leq M_k \|A_\varepsilon R_\varepsilon w - R_\varepsilon A_0 w\|_\varepsilon$$

where the constant M_k does not depend on ε .

A sketch of the proof of this theorem is given in [1], the full proof of theorem 1 and many of its applications to problems of mathematical physics are given in [2].

As an application of theorem 1, we consider the asymptotic behaviour of eigenvalues and eigenfunctions of the Dirichlet boundary value problem for a second order elliptic equation in a perforated domain with a periodic structure, which is characterized by a small parameter ε . This problem was posed by J.L.Lions [3] and considered by M. Vanninathan [4].

Let Ω^ε be a perforated domain of the form $\Omega^\varepsilon = \Omega \cap \varepsilon \omega$, where Ω is a smooth, bounded domain in \mathbb{R}^n , ω is a non bounded domain in \mathbb{R}^n , which is invariant with respect to shifts by vectors whose components are integers and where $\varepsilon \omega = \{x : \varepsilon^{-1} x \in \omega\}$. We assume that the boundary of ω is smooth and $0 < \varepsilon \leq 1$. Consider the family of elliptic operators

$$\mathcal{L}_\varepsilon(u) \equiv \frac{\partial}{\partial x_i} \left(a_{ij} \left(\frac{x}{\varepsilon} \right) \frac{\partial u}{\partial x_j} \right) - b \left(\frac{x}{\varepsilon} \right) u$$

where $a_{ij}(\xi)$, $b(\xi)$ are periodic smooth functions with period 1 in \mathbb{R}^n . We shall call such functions 1-periodic. Assume that the following conditions are satisfied :

$$\begin{aligned} \kappa_1 |\eta|^2 &\leq a_{ij}(\xi) \eta_i \eta_j \leq \kappa_2 |\eta|^2 & \kappa_1, \kappa_2 &= \text{const} > 0, \\ a_{ij}(\xi) &= a_{ji}(\xi) \quad i, j = 1, \dots, r, & b(\xi) &\geq 0 \end{aligned}$$

The summation over repeated indexes is assumed. We will characterize the asymptotics of eigenvalues and eigenfunctions as $\varepsilon \rightarrow 0$ for the eigenvalue problem :

$$(5) \quad \begin{cases} \mathcal{L}_\varepsilon(U_\varepsilon^k) + \Lambda_\varepsilon^k \rho \left(\frac{x}{\varepsilon} \right) U_\varepsilon^k = 0 & \text{in } \Omega^\varepsilon \\ U_\varepsilon^k \in H^1_0(\Omega^\varepsilon), k = 1, 2, \dots, & \text{and } \left(\rho \left(\frac{x}{\varepsilon} \right) U_\varepsilon^k, U_\varepsilon^j \right)_{L^2(\Omega^\varepsilon)} = \delta_{kj} \end{cases}$$

where $\rho(\xi)$ is a smooth, 1-periodic function in \mathbb{R}^n , $\rho(\xi) \geq C_0 = \text{const} > 0$, eigenvalues are numbered in such a way that $\Lambda_\varepsilon^k \leq \Lambda_\varepsilon^{k+1}$, $k = 1, 2, \dots$, and every eigenvalue is repeated as many times as its multiplicity. Here $H^1_0(\Omega^\varepsilon)$ is the Sobolev space, which consists of functions v , such that :

$$v \in L^2(\Omega^\varepsilon), \quad \frac{\partial v}{\partial x_j} \in L^2(\Omega^\varepsilon), \quad j = 1, \dots, n, \quad \text{and } v = 0 \text{ on } \partial\Omega^\varepsilon.$$

Let $\Phi(\xi)$ be an eigenfunction which corresponds to the first eigenvalue Λ_0 of the eigenvalue problem in domain ω with 1-periodic structure

$$(6) \quad \begin{cases} \frac{\partial}{\partial \xi_i} (a_{ij}(\xi) \frac{\partial \Phi}{\partial \xi_j}) + \Lambda_0 \rho(\xi) \Phi(\xi) = 0 & \text{in } \omega, \quad \Phi = 0 \text{ on } \partial\omega \\ \Phi(\xi) \text{ is 1-periodic in } \xi, \quad \text{and } \langle \rho(\xi) \Phi^2(\xi) \rangle = 1 \end{cases}$$

where $\langle f \rangle \equiv \int_{Q \cap \omega} f(\xi) d\xi$, $Q = \{x : 0 < x_j < 1, j = 1, \dots, n\}$.

As is well known, $\Phi(\xi)$ is a smooth function in ω , $\Phi \neq 0$ in ω and $|\nabla_{\xi} \Phi| \neq 0$ in a neighborhood of $\partial\omega$.

We represent eigenfunction U_{ε}^k of problem (5) in the form

$$U_{\varepsilon}^k(x) = \Phi\left(\frac{x}{\varepsilon}\right) v_{\varepsilon}^k(x)$$

It is easy to verify that v_{ε}^k , $k = 1, 2, \dots$, has to be an eigenfunction of the problem :

$$(7) \quad \begin{cases} M_{\varepsilon}(v_{\varepsilon}^k) + \lambda_{\varepsilon}^k \rho\left(\frac{x}{\varepsilon}\right) \Phi^2\left(\frac{x}{\varepsilon}\right) v_{\varepsilon}^k = 0 & \text{in } \Omega^{\varepsilon} \\ v_{\varepsilon}^k = 0 \text{ on } \partial\Omega^{\varepsilon} \cap \partial\Omega, \text{ and } (\rho\left(\frac{x}{\varepsilon}\right) \Phi^2\left(\frac{x}{\varepsilon}\right) v_{\varepsilon}^k(x), v_{\varepsilon}^j(x))_{L^2(\Omega^{\varepsilon})} = \delta_{kj} \end{cases}$$

where $\lambda_{\varepsilon}^k = \Lambda_{\varepsilon}^k - \Lambda_0 \varepsilon^{-2}$,

$$(8) \quad M_{\varepsilon}(u) \equiv \frac{\partial}{\partial x_i} \left(\Phi^2\left(\frac{x}{\varepsilon}\right) a_{ij}\left(\frac{x}{\varepsilon}\right) \frac{\partial u}{\partial x_j} \right) - \Phi^2\left(\frac{x}{\varepsilon}\right) b\left(\frac{x}{\varepsilon}\right) u$$

Thus, we obtain eigenvalue problem (7) for a second order elliptic equation which degenerates on the part of the boundary of the domain Ω^{ε} which is given by $\partial\Omega^{\varepsilon} \cap \Omega$.

By the introduction of weak solutions of the corresponding degenerate boundary value problem one can reduce the eigenvalue problem (7) to the eigenvalue problem for a compact, selfadjoint operator A_{ε} in the Hilbert space $\mathcal{H}_{\varepsilon}$. It can be proved that problem (7) has a discrete spectrum which is a non-decreasing sequence of eigenvalues $\lambda_{\varepsilon}^1, \lambda_{\varepsilon}^2, \dots$, where λ_{ε}^k are numbered with regard for the multiplicity in the same way as for problem (5).

If v_{ε}^k is an eigenfunction of problem (7) with eigenvalue λ_{ε}^k , then the function $\Phi\left(\frac{x}{\varepsilon}\right) v_{\varepsilon}^k(x)$ belongs to the space $H_1^0(\Omega^{\varepsilon})$ and it is an eigenfunction of

problem (5), corresponding to the eigenvalue

$$\Lambda_\varepsilon^k = \lambda_\varepsilon^k + \varepsilon^{-2} \Lambda_0$$

The theory of homogenization can be applied to the operators $M_\varepsilon(u)$. We consider now the boundary value problem for operators $M_\varepsilon(u)$.

$$(9) \quad M_\varepsilon(u^\varepsilon) = -\rho\left(\frac{x}{\varepsilon}\right) \Phi^2\left(\frac{x}{\varepsilon}\right) f^\varepsilon \text{ in } \Omega^\varepsilon, \quad u^\varepsilon = 0 \text{ on } \partial\Omega^\varepsilon \cap \partial\Omega$$

We note that there are no boundary conditions on the boundary $\partial\Omega^\varepsilon \cap \Omega$. We assume that $f^\varepsilon \in L^2(\Omega^\varepsilon)$ and that u^ε belongs to the space V^ε which consists of functions u such that :

$$\Phi\left(\frac{x}{\varepsilon}\right) u(x) \in L^2(\Omega^\varepsilon), \quad \Phi\left(\frac{x}{\varepsilon}\right) \frac{\partial u(x)}{\partial x_j} \in L^2(\Omega^\varepsilon), \quad j = 1, \dots, n$$

Using the Lax-Milgram theorem, one can prove the existence and the uniqueness of a solution of problem (9).

The homogenized problem for problem (9) has the form :

$$(10) \quad \bar{M}(u) = -\rho f^0 \text{ in } \Omega, \quad u \in H_0^1(\Omega)$$

where

$$(11) \quad \left\{ \begin{array}{l} \bar{M}(u) \equiv \bar{a}^{pq} \frac{\partial^2 u}{\partial x_p \partial x_q} - \bar{b} u, \quad \bar{\rho} = \theta \langle \Phi^2(\xi) \rho(\xi) \rangle \\ \bar{b} = \theta \langle \Phi^2(\xi) b(\xi) \rangle, \quad \theta = \langle \Phi^2(\xi) \rangle^{-1} \\ \bar{a}^{pq} = \theta \langle a_{pj} \Phi^2(\xi) \frac{\partial N_q(\xi)}{\partial \xi_j} + a_{pq}(\xi) \Phi^2(\xi) \rangle \end{array} \right.$$

and the functions $N_q(\xi)$ are solutions of the following boundary value problems :

$$(12) \quad \left\{ \begin{array}{l} \frac{\partial}{\partial \xi_k} (\Phi^2(\xi) a_{kj}(\xi) \frac{\partial N_q(\xi)}{\partial \xi_j}) = -\frac{\partial}{\partial \xi_k} (a_{kq}(\xi) \Phi^2(\xi)) \quad \text{in } \omega, \quad q = 1, \dots, n \\ N_q(\xi) \text{ is 1-periodic in } \xi, \quad \langle \Phi^2 N_q \rangle = 0 \\ \Phi \frac{\partial N_q}{\partial \xi_j} \in L^2(Q \cap \omega), \quad j = 1, \dots, n. \end{array} \right.$$

The operator \bar{M} is an elliptic operator with constant coefficients. The corresponding eigenvalue problem has the form

$$(13) \quad \bar{M}(v_o^k) + \lambda_o^k \rho v_o^k = 0 \text{ in } \Omega, \quad v_o^k \in H_0^1(\Omega), \quad (\rho v_o^\ell, v_o^m) = \delta_{\ell m}$$

where the eigenvalues λ_o^k are numbered with regard for the multiplicity and $\lambda_o^k \leq \lambda_o^{k+1}$.

THEOREM 2 : *Let Λ_ε^k , λ_ε^k , λ_o^k be the k -th eigenvalues of the eigenvalue problems (5), (7), (13) respectively. Then*

$$(14) \quad \Lambda_\varepsilon^k = \Lambda_o \varepsilon^{-2} + \lambda_\varepsilon^k, \quad |\lambda_\varepsilon^k - \lambda_o^k| \leq C_k \varepsilon$$

where the constant C_k does not depend on ε .

If the multiplicity of the eigenvalue $\lambda_o = \lambda_o^{\ell+1}$ is equal to m , i.e., $\lambda_o^{\ell+1} = \dots = \lambda_o^{\ell+m}$, and $v_o(x)$ is an eigenfunction of problem (13) with eigenvalue λ_o , then for all $\varepsilon \in]0, 1]$ there exists a function v^ε such that

$$(15) \quad \|(v^\varepsilon - v_o) \Phi\left(\frac{x}{\varepsilon}\right)\|_{L^2(\Omega^\varepsilon)} \leq M_\ell \varepsilon$$

where v^ε is a linear combination of eigenfunctions of the eigenvalue problem (7), corresponding to eigenvalues $\lambda_\varepsilon^{\ell+1}, \dots, \lambda_\varepsilon^{\ell+m}$.

In order to prove theorem 2 one has to introduce suitable spaces \mathcal{H}_ε and \mathcal{H}_o and operators A_ε , A_o such that $A_\varepsilon f^\varepsilon = u^\varepsilon$, $A_o f^o = u$ and u^ε , u are solutions of boundary value problems (9), (10) respectively. One can check that conditions $C_1 - C_4$ are satisfied and therefore theorem 1 is applicable. The eigenvalues of problems (7), (13) and problems (1), (2) are connected by the relations :

$$(\lambda_\varepsilon^k)^{-1} = \mu_\varepsilon^k, \quad (\lambda_o^k)^{-1} = \mu_o^k$$

Using methods of the theory of homogenization, one can estimate the right-hand side of (3) and prove that it is of order ε . Therefore, estimates (14), (15) are consequences of (3), (4).

The full proof of theorem 2 is given in [2].

Let us consider now another example of the application of theorem 1. We shall study asymptotics of vibrating systems with concentrated masses. These problems were considered in [5] - [11].

Let Ω be a smooth, bounded domain, $\Omega \subset \mathbb{R}^n$, $n \geq 3$, $O \in \Omega$ and O is an origin. Consider the eigenvalue problem:

$$(16) \quad \begin{cases} \Delta u_\varepsilon^k + \lambda_\varepsilon^k (1 + \varepsilon^{-m} \chi(\frac{x}{\varepsilon})) u_\varepsilon^k = 0 & \text{in } \Omega, \quad u_\varepsilon^k = 0 \quad \text{on } \partial\Omega \\ ((1 + \varepsilon^{-m} \chi(\frac{x}{\varepsilon})) u_\varepsilon^k, u_\varepsilon^j)_{L^2(\Omega)} = \delta_{kj} \end{cases}$$

where $\varepsilon \in]0, 1[$, λ_ε^j are numbered with regard for the multiplicity and $\lambda_\varepsilon^k \leq \lambda_\varepsilon^{k+1}$. Here $\chi(\xi)$ is a bounded measurable function such that $\chi(\xi) > M = \text{const} > 0$ for $\xi \in G$ and $\chi(\xi) = 0$ for $\xi \notin G$, where G is an open set of positive measure such that $\bar{G} \subset \Omega$, $O \in G$.

THEOREM 3 : I – Assume that $-\infty < m < 2$. Then

$$|\lambda_\varepsilon^k - \lambda_0^k| \leq C_k \varepsilon^{1-m+n/2}$$

where the constant C_k does not depend on ε and λ_0^k is the k -th eigenvalue of the Dirichlet eigenvalue problem in Ω

$$(17) \quad \Delta u_0^k + \lambda_0^k u_0^k = 0, \quad u_0^k = 0 \quad \text{on } \partial\Omega$$

II – Assume that $m > 2$. Then

$$\lambda_\varepsilon^k = \Lambda_0^k \varepsilon^{m-2} + \varepsilon^{m-2} \alpha_\varepsilon^k$$

where Λ_0^k is the k -th eigenvalue of the eigenvalue problem

$$\Delta_\xi u_0^k + \Lambda_0^k \chi(\xi) u_0^k = 0, \quad \xi \in \mathbb{R}^n, \quad u_0^k \in H$$

Here H is the completion of $C_0^\infty(\mathbb{R}^n)$ -functions with the norm

$$\|u\|_H = \|\nabla_\xi u\|_{L^2(\mathbb{R}^n)} \\ \alpha_\varepsilon^k = C_k (\varepsilon^{n-2} + \varepsilon^{(m-\gamma_n)/2}),$$

$\gamma_3 = 1, \gamma_4 = \text{const} \in]0, 1]$, $\gamma_n = 0$ for $n > 4$ and the constant C_k does not depend on ε .

III – Assume that $m = 2$. Then

$$|\lambda_\varepsilon^k - \Lambda^k| \leq C_k \varepsilon^{(2-\gamma_n)/2}$$

where the constant C_k does not depend on ε . Here γ_n is defined as above and Λ^k is the k -th eigenvalue of the eigenvalue problem for the system

$$\Delta_\xi U_0^k(\xi) + \Lambda^k \chi(\xi) U_0^k(\xi) = 0 \quad \text{in } \mathbb{R}^n, \quad U_0^k \in H \\ \Delta_x u_0^k(x) + \Lambda^k u_0^k(x) = 0, \quad u_0^k \in H_0^1(\Omega).$$

In order to prove theorem 3 in the case $-\infty < m < 2$, $n \geq 3$, we introduce spaces \mathcal{K}_ε , \mathcal{K}_0 and operators A_ε , A_0 in such a way that conditions $C_1 - C_4$ of theorem 1 are satisfied. Let \mathcal{K}_ε and \mathcal{K}_0 be $L^2(\Omega)$ spaces with the scalar products :

$$(f^\varepsilon, g^\varepsilon)_{\mathcal{K}_\varepsilon} \equiv \int_{\Omega} (1 + \varepsilon^{-m} \chi(\frac{x}{\varepsilon})) f^\varepsilon \cdot g^\varepsilon dx \quad (f^0, g^0)_{\mathcal{K}_0} \equiv \int_{\Omega} f^0 \cdot g^0 dx$$

respectively. We set :

$$R_\varepsilon f^0 = (1 - \mathfrak{X}_G(\frac{x}{\varepsilon})) f^0 \quad f^0 \in \mathcal{K}_0$$

where $\mathfrak{X}_G(\xi)$ is the characteristic function of the set G : $\mathfrak{X}_G(\xi) = 1$ for $\xi \in G$, $\mathfrak{X}_G(\xi) = 0$ for $\xi \notin G$.

We define the operator $A_\varepsilon : \mathcal{K}_\varepsilon \rightarrow \mathcal{K}_\varepsilon$ in the following way : for $f^\varepsilon \in \mathcal{K}_\varepsilon$

$$A_\varepsilon f^\varepsilon = u^\varepsilon$$

where u^ε is a solution of the Dirichlet problem

$$\Delta_x u^\varepsilon = -(1 + \varepsilon^{-m} \chi(\frac{x}{\varepsilon})) f^\varepsilon \quad \text{in } \Omega, \quad u^\varepsilon \in H_0^1(\Omega)$$

We denote by A_0 an operator $A_0 : \mathcal{K}_0 \rightarrow \mathcal{K}_0$ such that for any $f^0 \in \mathcal{K}_0$ we have :

$$A_0 f^0 = u^0$$

where u^0 is a solution of the Dirichlet problem

$$\Delta u^0 = -f^0(x) \quad \text{in } \Omega, \quad u^0 \in H_0^1(\Omega).$$

It is easy to verify that conditions $C_1 - C_4$ are satisfied.

Consider the case $m > 2$, $n \geq 3$. Let u^0 be a solution of the problem

$$(18) \quad \Delta_\xi u^0 = -\chi(\xi) f^0 \quad \text{in } \mathbb{R}^n, \quad f^0 \in L_{loc}^2(\mathbb{R}^n), \quad u^0 \in H$$

We define the space \mathcal{K}_0 as $L^2(G)$ with the scalar product

$$(f^0, g^0)_{\mathcal{K}_0} = \int_G \chi(\xi) f^0 g^0 d\xi$$

We set $A_0 f^0 = \mathfrak{X}_G(\xi) u^0$, where u^0 is a solution of problem (18). It can be proved that A_0 is a positive, compact, selfadjoint operator, $A_0 : \mathcal{K}_0 \rightarrow \mathcal{K}_0$.

The space \mathcal{K}_ε is defined as $L^2(\Omega_\varepsilon)$ with the scalar product

$$(f^\varepsilon, g^\varepsilon)_{\mathcal{K}_\varepsilon} = \int_{\Omega_\varepsilon} (\varepsilon^m + \chi(\xi)) f^\varepsilon g^\varepsilon d\xi$$

where $\Omega_\varepsilon = \varepsilon^{-1} \Omega$ and $\varepsilon^{-1} \Omega = \{x : \varepsilon x \in \Omega\}$. The operator $R_\varepsilon : \mathcal{K}_0 \rightarrow \mathcal{K}_\varepsilon$ is an extension of $f^0 \in L^2(G)$ such that $f^0 = 0$ in $\Omega_\varepsilon \setminus G$. We define $A_\varepsilon : \mathcal{K}_\varepsilon \rightarrow \mathcal{K}_\varepsilon$ by

setting :

$$A_\varepsilon f^\varepsilon = u^\varepsilon$$

where u^ε is a solution of the boundary value problem :

$$\Delta_\xi u^\varepsilon = -(\varepsilon^m + \chi(\xi))f^\varepsilon \quad \text{in } \Omega_\varepsilon, \quad u^\varepsilon \in H_0^1(\Omega_\varepsilon).$$

For $\mathcal{K}_\varepsilon, \mathcal{K}_0, A_\varepsilon, A_0$ defined as above, conditions C_1-C_4 of theorem 1 are satisfied and theorem 3 in the case $m > 2$ is also a consequence of theorem 1.

In a similar way the case $m = 2$ can be considered. The detailed proof of theorem 3 is given in [2].

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