

# Necessary conditions for strong hyperbolicity of first order systems

by

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## §0. Introduction, definitions and theorems.

On higher order scalar equations, the strong hyperbolicity is well characterized. (See O. A. Oleinik [13], V. Ja. Ivrii and V. M. Petkov [3], V. Ja. Ivrii [2], L. Hörmander [1], N. Iwasaki [4], [5], [6], etc.) On the other hand, on first order systems, if their coefficients are constant, we also have a complete result. (See K. Kasahara and M. Yamaguti [7]). In case of first order systems with variable coefficients, we have some results, but they are not satisfactory. (See, for example, N. D. Koutev and V. M. Petkov [8], T. Nishitani [10], [11], [12], H. Yamahara [14], [15] etc.).

In this note, we give some necessary conditions for the strong hyperbolicity of first order systems with variable coefficients, assuming that coefficients depend only on the time variable. This is a further developed results of H. Yamahara [14] and [15]. On the other hand, these become sufficient under a reasonable supplementary condition.

Let us consider the following Cauchy problem.

$$(1) \quad \begin{cases} P u \equiv (P_p - B)u \equiv \{D_t - \sum_{i=1}^{\ell} A_i(t, x) D_{x_i} - B(t, x)\}u = f(t, x), \\ u(t_0, x) = u_0(x), \end{cases}$$

where  $u(t, x)$ ,  $u_0(x)$ ,  $f(t, x)$  are vectors of dimension  $N$  and  $A_i(t, x)$ ,  $B(t, x)$  are square matrices of order  $N$  with elements in  $C^\infty(\Omega)$ , ( $\Omega$  is an open set in  $\mathbb{R}_{t, x}^{1+\ell}$ ). We say

that the Cauchy problem (1) is uniformly well-posed in  $\Omega$  if the following holds :

$$(2) \quad \begin{cases} \forall K = [T_1, T_2] \times K_0, \forall K' \subset \subset K \\ \exists \omega : \text{a lens-shaped neighborhood of the origin,} \\ \forall (t_0, x_0) \in K', \forall u_0 \in C^\infty(K_0), \forall f \in C^\infty(K), \exists ! u \text{ solution of (1) in } (t_0, x_0) + \omega. \end{cases}$$

**Proposition 0.1.** *If (1) is uniformly well-posed in  $\Omega$ , the following holds :*

$$(3) \quad \begin{cases} \forall (\hat{t}_0, \hat{x}_0) \in K', \forall M \in \mathbb{N}, \exists M' \in \mathbb{N}, \exists \delta > 0, \exists C > 0 \\ \forall (t_0, x_0) \in K' \text{ s.t. } |t_0 - \hat{t}_0| \leq \delta, \forall U_0 \in C^M(K) \\ \forall f \in C^{M-1}(K), \exists u \text{ solution of (1) in } K'', (K'' = \{|t - t_0| \leq \delta\} \times \{|x - x_0| \leq \delta\}) \end{cases}$$

and  $u$  satisfies

$$(4) \quad |u|_{M, K''} \leq C(|u_0|_{M', K_0} + |f|_{M'-1, K}),$$

$$\text{where } |u|_{M, K} \stackrel{\text{def}}{=} \sum_{|\alpha| \leq M} \max_{(t, x) \in K} |D_{t, x}^\alpha u(t, x)|.$$

By the estimate (4), we have the following theorem.

**Theorem 0.** *(P. D. Lax and S. Mizohata)*

*If (1) is uniformly well posed, all characteristic roots of  $P_p$  are real in  $\Omega \times \mathbb{R}_\xi^\ell \setminus 0$ .*

From now on, we always suppose the conclusion of the above theorem.

**Definition 0.1.** (Strong hyperbolicity)

We say that  $P_p$  is strongly hyperbolic when the Cauchy problem (1) of  $P_p + B$  is uniformly well posed in  $\Omega$  for arbitrary choice of  $B(t, x)$ .

Throughout this note, we assume the following :

**Assumption.**  $A_i$  depends only on  $t$ , ( $1 \leq i \leq \ell$ ).

Let  $\{\lambda_j\}_{j=1}^d$  be the different characteristic roots of  $P_p$  at  $t = t_0$  and  $\xi = \xi_0 \neq 0$ .

We set

$$A^{(0)} = \sum_{i=1}^{\ell} A_i(t_0) \xi_{0i},$$

$$A^{(i)} = \sum_{i=1}^{\ell} \frac{\partial}{\partial t} A_i(t_0) \xi_{0i},$$

$\mathcal{P}_j$  : the projection to the generalized eigenspace of  $\lambda_j$ ,

$$A_j^{(i)} = A^{(i)} \mathcal{P}_j, \quad (0 \leq i \leq 1, 1 \leq j \leq d).$$

**Theorem 1.** *If  $P_p$  is strongly hyperbolic in  $\Omega$ , the following holds*

$$(5) \quad \mathcal{P}_j (A_j^{(0)} - \lambda_j I_N) (A_j^{(i)})^k (A_j^{(0)} - \lambda_j I_N) = 0$$

for  $1 \leq j \leq d$  and  $k \in \mathbb{Z}_+ = \{0, 1, \dots\}$ .

**Remark.** Let  $m^j$  be the multiplicity of  $\lambda_j$ . At least for  $k \geq m^j$ , Condition (5) becomes trivial.

**Corollary 2.** *The lengths of Jordan chains of  $A^{(0)}$  are at most 2.*

By virtue of Bronshtein–Mandai's theorem, the characteristic roots  $\lambda^{(j)}(t)$  ( $1 \leq j \leq N$ ) of  $P_p(t_i; \xi_0)$  belong to  $C_t^\infty$ . (See T. Mandai [17] and M. D. Bronshtein [16]). Let us set

$$\lambda_0^{(j)}(t) = \lambda^{(j)}(t_0) + (t - t_0) \frac{\partial}{\partial t} \lambda^{(j)}(t_0).$$

**Theorem 3.** *If  $\ell = 1$  and  $\{\lambda_0^{(j)}(t)\}_{j=1}^N$  are distinct for  $0 < |t - t_0| \leq \exists \delta_0$ , condition (5) is sufficient for the strong hyperbolicity of  $P_p$  near  $t_0$ .*

**Remark.**  $\lambda_0^{(j)}(t)$  is obtained by  $\sum_i \left(\frac{\partial}{\partial t}\right)^k A_i(t_0) \xi_{0i}$  with  $0 \leq k \leq 2$ .

In the following sections 1, 2 and 3, we give a proof of Theorem 1 for  $k = 0$  and 1. The proof of Theorem 1 for  $k \geq 2$  and that of Theorem 3 will be given in the forthcoming paper [19].

## §1. Reduction.

We may assume  $t_0 = 0$ . We take  $B$  as constant matrix and  $f = 0$ . Let us take Fourier image of (1) on the variable  $x$ ;

$$(1.1) \quad \begin{cases} \{D_t - \sum_{i=0}^{\ell} A_i(t) \xi_i - B(t)\} \hat{u} = 0 \\ \hat{u}(0, \xi) = \hat{u}_0(\xi). \end{cases}$$

Setting  $\xi = n\xi_0$ , we expand  $\sum_{i=0}^{\ell} A_i(t) \xi_{0_i}$  as  $A^{(0)} + t A^{(1)} + t^2 A^{(2)}(t)$ . Further, we transform  $A^{(0)}$  to Jordan's normal form  $\Lambda$ :

$$\Lambda = \begin{pmatrix} \Lambda_1 & & \\ & \ddots & \\ & & \Lambda_d \end{pmatrix}, \quad \Lambda_j = \lambda_j I_{m_j} + J^j,$$

$$J^j = J^j(r_j, 1) \oplus J^j(r_j, 2) \oplus \dots \oplus J^j(r_j, m_{r_j}^j) \oplus \dots \oplus J^j(1, m_1^j),$$

$$J^j(k, h) = \begin{pmatrix} 0 & 1 & & \\ & \cdot & \cdot & \\ & & \cdot & \\ & & & 1 \\ & & & & 0 \end{pmatrix}; \quad k \times k, \quad (1 \leq j \leq d).$$

Thus, we arrive at

$$(1.2) \quad \begin{cases} \{D_t - n(\Lambda + t \tilde{A}^{(1)} + t^2 \tilde{A}^{(2)}) - \tilde{B}(t)\} \hat{u}_1 = 0, \\ \hat{u}_1(0) = \hat{u}_{10}. \end{cases}$$

Corresponding to  $\Lambda$ , we can transform (1.2) by the similar transformation by  $N(t) = I + t N_1$ :

$$(1.3) \quad \begin{cases} \hat{P} \hat{u} = \{D_t - n(\Lambda + t \tilde{A}^{(1)} + t^2 \tilde{A}^{(2)}(t)) - \tilde{B}(t)\} \hat{u}_2 = 0, \\ \hat{u}_2(0) = \hat{u}_{x_0}. \end{cases}$$

where, decomposing in blocks  $\tilde{A}^{(1)}$  and  $\tilde{A}^{(2)}$  corresponding to  $\Lambda$ , say,  $(\tilde{A}^{(1)}(j, j'))_{1 \leq j, j' \leq d}$  and  $(\tilde{A}^{(2)}(j, j'))_{1 \leq j, j' \leq d}$ , it holds that  $\tilde{A}(j, j) = \tilde{A}(j, j)$  and  $\tilde{A}(j, j') = 0$  for  $j \neq j'$ .

Ex.

$$\Lambda = \begin{matrix} \Lambda_1 & & & \\ & \ddots & & \\ & & \Lambda_2 & \\ & & & \ddots \end{matrix}, \quad A = \begin{matrix} A(1,1) & A(1,2) & & \\ & \ddots & \ddots & \\ & & A(2,1) & A(2,2) \\ & & & \ddots \end{matrix}$$

As our consideration becomes independent of the part which has the factor  $t^2 n$ , from now on, we take out  $(j,j)$  block and omit the subscript "j". We may assume  $\lambda = 0$ . Further, we set  $t = n^{-\sigma} s$  ( $\sigma > 0$ ). Thus, we arrive at

$$(1.4) \quad \begin{cases} P_0 v \equiv \{n^\sigma D_s - (n J + n^{1-\sigma} s A_1 + n^{1-2\sigma} s^2 A_2(s) + B)\}v \\ = 0, \\ v(0) = v_0, \end{cases}$$

where  $v, v_0$  are vectors of dimension  $m$ ,  $J, A_1, A_2, B$  are square matrix of order  $m$  and

$$J = J(r,1) \oplus \dots \oplus J(r,m_r) \oplus J(r-1,1) \oplus \dots \oplus J(r-1,m_{r-1}) \oplus J(r-2,1) \oplus \dots \oplus J(1,m_1),$$

$$\sum_{j=1}^r j m_j = m.$$

Here, condition (5) for  $j$  in  $\mathbb{S}_0$  is equivalent to

$$(1.5) \quad J(A_1)^k J = 0 \text{ for } k \in \mathbb{Z}_+.$$

**Proposition 1.1.** *We assume that (1) is uniformly well posed in  $\Omega$ . If, for  $P_0$  in (1.4), there exists an invertible matrix  $N(s,n)$  for  $0 < |s| \leq \delta$  and  $\ell \geq 2$ , ( $\ell \in \mathbb{N}$ ) such that*

$$\tilde{P} = N^{-1} L N = n^\sigma D_s - n^\mu (\tilde{J}(s) + \tilde{K}(s)) - n^{\mu'} C(s,n),$$

$\mu > \mu', \mu > \sigma$ ,  $C(s,n)$  is bounded,

$$\tilde{J} = \bigoplus_{\substack{1 \leq k \leq R \\ 1 \leq h \leq M_R}} \tilde{J}(k,h), \quad \tilde{J}(k,h) = \begin{pmatrix} 0 & a_1^{k,h} & & \\ & 0 & & \\ & & & a_{k-1}^{k,h} \\ & & & 0 \end{pmatrix},$$

$a_1^{k,h}$  is not identically zero, and is analytic for  $s \neq 0$ ,

$\tilde{K} = (K(k,h,k',h'))_{\substack{1 \leq k,k' \leq R \\ 1 \leq h \leq M_k \\ 1 \leq h' \leq M_{k'}}}$ : block decomposed with respect to  $\tilde{J}$ ,

with

$$K(k,h,k',h') = \begin{pmatrix} k,h,k',h' & & & \\ \alpha_1 & & & 0 \\ & & & 0 \\ & & k,h,k',h' & \\ \alpha_k & & & 0 \end{pmatrix} ; \quad k \times k'$$

$$\alpha_i^{k,h,k',h'} = 0 \text{ for } i \neq 0 \pmod{\ell},$$

then, we have the following ;

- 1) If  $\ell \geq 3$ ,  $\tilde{J} + \tilde{K}$  is nilpotent.
- 2) Let  $\det(\lambda I - (\tilde{J} + \tilde{K}))$  be  $\sum_{i=0}^m C_i(s) \lambda^{m-i}$ .  
If  $\ell = 2$  and  $C_{2i}(s)$  ( $C_{2i+1}(s)$ , resp.) is even function (odd function, resp.),  $\tilde{J} + \tilde{K}$  is nilpotent.

Now, we assume (5) does not hold for  $k = 0$  or  $k = 1$ .

In order to make B stronger than  $n^{1-2\sigma} s^2 A_2(s)$ , we take  $1-2\sigma < 0$  ie  $\sigma > \frac{1}{2}$ .

## §2. Maximal connection.

Let us consider

$$\tilde{J} = \bigoplus_{\substack{1 \leq k \leq R \\ 1 \leq h \leq M_R}} \tilde{J}(k,h), \quad \tilde{J}(k,h) \text{ is that in Prop. 1.1, } M = \sum_{j=1}^R j M_j.$$

Corresponding to the blocks of  $\tilde{J}$ , we decompose  $M \times M$  matrix  $K$  to  $(K(k,h,k',h'))_{\substack{1 \leq k,k' \leq R \\ 1 \leq h \leq M_k \\ 1 \leq h' \leq M_{k'}}}$ .

**Ex.**

$$\tilde{J} = \begin{pmatrix} \tilde{J}(2,1) & & \\ & \tilde{J}(2,2) & \\ & & \tilde{J}(1,1) \end{pmatrix}, \quad K = \begin{pmatrix} \tilde{J}(2,1,2,1) & \tilde{J}(2,1,2,2) & \tilde{J}(2,1,1,1) \\ \tilde{J}(2,2,2,1) & \tilde{J}(2,2,2,2) & \tilde{J}(2,2,1,1) \\ \tilde{J}(1,1,2,1) & \tilde{J}(1,1,2,2) & \tilde{J}(1,1,1,1) \end{pmatrix}$$

We call  $(K(k,h,k',h'))$  the block decomposition of  $K$  with respect to  $\tilde{J}$ .

The following notions are important.

**Definition 2.1.** (Maximal connection of Jordan chain).

Let  $(K(k,h,k',h'))$  be the block decomposition of  $K$  with respect to  $\tilde{J}$ . If

1)

$$(2.1) \quad \begin{cases} K(R,h,k',h') = \begin{pmatrix} 0 & & 0 \\ \cdot & & \\ 0 & & 0 \\ \alpha^{R,h,k',h'} & & 0 \end{pmatrix}, \\ K(k,h,k',h') = 0 \quad (k < R) \end{cases}$$

for arbitrary  $h, k'$  and  $h'$ ,

$$(2) \quad K \neq 0,$$

$$(3) \quad \mathcal{A} = (\alpha^{R,h,R,h'})_{1 \leq h, h' \leq M_R} \text{ is nilpotent,}$$

$\tilde{J}+K$  is again nilpotent. We say that in  $\tilde{J}+K$ , the Jordan chains of  $\tilde{J}$  are maximally connected by  $K$ , or that  $K$  brings a maximal connection (of Jordan chain) to  $\tilde{J}$ .

**Definition 2.2.** (Selfsimilar matrix).

Let us take  $1 < R_0 < R_1 < \dots < R_p < R_{p+1}$ , such that

$$R_{j+1} = k_j R_j + R_j^0, \quad k_j \geq 1, \quad 0 \leq R_j^0 < R_j, \quad k_j, R_j \in \mathbb{N}.$$

We set

$$A_0 = \begin{pmatrix} 0 & 1 & & \\ & 0 & \cdot & \\ & & \cdot & 1 \\ & & & 0 \end{pmatrix}; \quad R_0 \times R_0,$$

$$A_{j+1} = \underbrace{A_j \oplus \dots \oplus A_j}_{k_j} \oplus A_j^0 + K_j; \quad R_{j+1} \times R_{j+1}$$

$A_j^0$  = the first  $R_j^0$  rows and  $R_j^0$  columns part of  $A_j$ ,

$$(2.2) \quad K_j = (K_j(k,k')); \text{ block decomposition w.r.t. } A_{j+1}$$

$$K_j(h,h+1) = \begin{pmatrix} 0 & 0 \\ \cdot & 0 \\ 0 & \\ \vdots & \\ s & 0 \end{pmatrix}, \quad 1 \leq h \leq k_j,$$

$\hat{J}(i) : i \times i$  = the first  $i$  rows and  $i$  columns part of  $A_{p+1}$ .

We call

$$\hat{J} = \bigoplus_{1 \leq i \leq R_{p+1}} (J(i) \otimes \dots \otimes J(i))_{M_i}$$

a self similar matrix of step  $p+1$  and  $A_j$  and  $A_j^0$  the factors of step  $j$ .

Let  $\hat{J}$  be  $M \times M$  selfsimilar matrix and  $K$  is a  $M \times M$  matrix block decomposed w.r.t  $\hat{J}$ . Let an element of a block of  $K$  belong to  $q_p^{(r)}$ -th  $A_p$  in the direction of row and to  $q_p^{(c)}$ -th  $A_p$  in the direction of column. ( $(k_p+1)$ -th  $A_p = A_p^0$ ). Further, in  $A_p$ , let it belong to  $q_{p-1}^{(r)}$ -th  $A_{p-1}$  in the direction of row and to  $q_{p-1}^{(c)}$ -th  $A_{p-1}$  in the direction of column. We continue this procedure up to  $q_0^{(*)}$ . At last, let it be the  $(q_{-1}^{(r)}, q_{-1}^{(c)})$  element of  $A_0$ . We set  $q_h = q_h^{(r)} - q_h^{(c)} + 1$  ( $-1 \leq h \leq p$ ).

**Definition 2.3. (Address)**

We call  $q = (q_p, q_{p-1}, \dots, q_{-1})$  the address of the element.

To the set of addresses, we give the dictionary order.

**Definition 2.4. (Acceptable matrix).**

Let us take  $1 > v_{-1} > v_0 > \dots > v_p > 0$ ,  $v = \sum_{j=-1}^p v_j$ , and  $\sigma > 0$  ( $v_j, \sigma \in \mathbb{R}_+$ ). For a block decomposed matrix  $K$  w.r.t. a selfsimilar matrix  $\hat{J}$ , if the address  $q$  of its element has a  $q_j$  such that  $q_j = k_j + 1$  and  $\sum_{h=0}^{j-1} (q_h - 1)R_h + q_{-1} = R_j^0$  (that is, the element is found at the left-down corner of  $R_{j+1} \times R$  matrix in step of  $j+1$ ,  $R(\leq R_{j+1})$ ; free), the element has the form  $c(s)n^{1-v'}$ ,  $v' = v'(q) = 2\sigma - \sum_{j=-1}^p (q_j - 1)v_j$  and otherwise, it has the form  $c n^{1-v'}$ ,  $v' = v'(q) = \sigma - \sum_{j=-1}^p (q_j - 1)v_j$  and  $c$  is constant. Further, if all  $v(q)$  are greater than  $v$ , we say that  $K$  is acceptable w.r.t.  $n^{1-v}\hat{J}$ . We call  $\varepsilon(q) = (v' - v) / (\sum_{j=0}^p (q_j - 1)R_j + q_{-1})$  the descent index of the element with the address  $q$ .

When the descent index is smaller, we say that it is more effective.

**Remark.** Corresponding to the above  $\hat{J}$ , we take a shearing operator with weight  $\varepsilon$

$$W = \bigoplus_{\substack{1 \leq k \leq R_{p+1} \\ 1 \leq h \leq M_k}} W(k, h, \varepsilon),$$

$W(k, h, \varepsilon) = \text{diag}(1, n^\varepsilon, n^{2\varepsilon}, \dots, n^{(k-1)\varepsilon})$ ,  $\varepsilon = v(q)$ . Then, the element with the address  $q$



obtain the order  $1-\nu-\varepsilon$  of  $W^{-1}(n^{1-\nu} \hat{J})W$  by the shearing transformation  $W^{-1} K W$ .

Now, we return to the equation (1.4). We assume that (1) is uniformly well-posed.

$$\text{Let us set } W = \bigoplus_{\substack{1 \leq k \leq r \\ 1 \leq h \leq m_k}} W(k, h, \frac{\sigma}{r}).$$

setting  $w_1 = W^{-1} w$ ,  $w_1$  satisfies  $P_1 w_1 = 0$ ,

$$(2.3) \quad P_1 = W^{-1} P_0 W = n^\sigma D_s - (n^{1-\sigma/r} J + n^{1-\sigma/r} s K_1 + s A_1^1(n) + s^2 A_2^1(s; n) + B^1(n)),$$

where  $s(n^{1-\sigma/r} K_1 + A_1^1(n))$  is brought from  $n^{1-\sigma} s A^1$  and the order of  $A_1^1(n)$  is less than  $1-\sigma/r$ .  $s A_1^1(n) + s^2 A_2^1(s; n)$  is acceptable w.r.t.  $n^{1-\sigma/r} J$ .

$$\text{In } K_1 = (K_1(k, h, k', h'))_{\substack{1 \leq k, k' \leq r \\ 1 \leq h \leq m_k \\ 1 \leq h' \leq m_{k'}}}, \quad K_1(k, k', h') = \begin{pmatrix} 0 & & & \\ & \cdot & & \\ & & 0 & \\ \alpha^{h, k', h'} & & & 0 \end{pmatrix}$$

and  $K_1(k, h, k', h') = 0$  for  $k < r$ . By virtue of Proposition 1.1,  $(\alpha^{h, R, h'})_{1 \leq h, h' \leq m_r}$  must be

nilpotent, and then,  $K_1$  brings a maximal connection to  $J$  if  $K_1 \neq 0$ . We can take each Jordan chain in  $J + K_1$  composed by vectors of  $s^\mu v$ ,  $v$ : constant vector. Replacing  $s^\mu v$  by  $vn$  we can have a constant matrix  $N$  which transform  $J$  to  $\hat{J}_1$ , a selfsimilar matrix. We have

$$(2.4) \quad \tilde{P}_1 = N^{-1} P_1 N = n^\sigma D_s - (n^{1-\sigma/r} \hat{J}_1(s) + s \tilde{A}_1^1(n) + s^2 \tilde{A}_2^1(s; n) + \tilde{B}^1(n)).$$

Let us set the length of the longest Jordan chain of  $\hat{J}_1(s)$  as  $R_1 = k_0 R_0 + \ell$ ,  $R_0 = r$ ,  $0 \leq \ell < R_0$ . In  $s \tilde{A}_1^1(n) + s^2 \tilde{A}_2^1(s; n)$ , the highest order on  $n$  is given only by the elements with the address  $(k_0 + 1, r - 1)$  if  $\ell \geq R_0 - 1$ , and by those with the address  $(k_0, r - 1)$  (and also by those with  $(k_0 + 1, r - 2)$  in case of  $k_0 = 1$ ) if  $\ell < R_0 - 1$ . In the former case, if an element with the address  $(k_0 + 1, r - 1)$  does not vanish, after the shearing transformation with weight  $\frac{\sigma}{R_0 R_1}$ , a maximal connection occurs by virtue of Proposition 1.1. In the latter case, no maximal connection occurs. Continuing this procedure, we arrive at the following proposition.

**Proposition 2.1.** Let us set  $R_{-1} = 1$ ,  $R_0 = R$ ,  $R_{j+1} = k_j R_j + R_j - R_{j-1}$  ( $0 \leq j \leq p+1$ ) and  $\hat{R} = k_p R_p + \ell$ , ( $k_j \in \mathbb{N} = \{1, 2, \dots\}$ ,  $0 \leq \ell < R_p$ ).

- (1) In the above procedure, if  $p$  times maximal connections occur, the highest order part must be the selfsimilar matrix of step  $p+1$  replacing  $R_{p+1}$  by  $\hat{R}$  and has the order  $1-\nu$  on  $n$ ,  $\nu = \sum_{j=0}^p \nu_j$ ,  $\nu_j = \frac{\sigma}{R_{j-1} R_j}$ .
- (2) The operator  $\tilde{P}_{p+1}$  has the following form ;
- (2.5)  $\tilde{P}_{p+1} = n^\sigma D_s - (n^{1-\nu} \tilde{J}_{p+1}(s) + s A_1^{p+1}(n) + s^2 A_2^{p+1}(s; n) + B^{p+1})$ ,  
 where  $s A_1^{p+1}(n) + s^2 A_2^{p+1}(s; n)$  is acceptable w.r.t.  $n^{1-\nu} \tilde{J}_{p+1}$ .
- (3) In  $s A_1^{p+1}(n) + s^2 A_2^{p+1}(s; n)$ , if  $\ell \geq R_p - R_{p-1}$ , the highest order is given only by the elements with address  $(k_p+1, k_{p-1}, \dots, k_0, r-1)$  and if  $\ell < R_p - R_{p-1}$ , it is given by those with the address  $(k_p, k_{p-1}, \dots, k_0, r-1)$  (and also by those with  $(1, \dots, 1, 2, k_\ell-1, k_{\ell-1}, \dots, k_0, r-1)$  in case of  $k_{\ell+1} = \dots = k_p = 1$  and  $k_\ell \geq 2$  and also by those with  $(1, \dots, 1, 2, 1, \dots, 1, r-2)$  in case of  $k_0 = k_1 = \dots = k_p = 1$ ).

Proof By the induction on  $p$ .

### §3 Proof of Theorem 1, case of $k \leq 1$ .

The maximal connections can occur at most  $\lfloor \frac{m-r}{r-1} \rfloor$  times. Let no maximal connection occur on  $\tilde{P}_{p+1}$ , that is, in  $\hat{R} = k_p R_p + \ell$ ,  $\ell \neq R_p - R_{p-1}$  or  $\ell = R_p - R_{p-1}$  but all elements with the address  $(k_p+1, k_{p-1}, \dots, k_0, r-1)$  vanish.

Let  $W$  be the shearing operator corresponding to  $\tilde{J}_{p+1}$  in (2.5) with weight  $\varepsilon$  ( $\varepsilon = \frac{\sigma}{R_p R_{p+1}}$  in case of  $\ell > R_p - R_{p-1}$  and  $\varepsilon = \frac{\sigma}{R_p (R_{p+1} - R_p)}$  in case of  $\ell \leq R_p - R_{p-1}$ ).

We set

$$(3.1) \quad \hat{P}_{p+2} \equiv W^{-1} \tilde{P}_{p+1} W = n^\sigma D_s - \{n^{1-\nu-\varepsilon} (\tilde{J}_{p+1}(s) + s K_{p+2}) + s A_1^{p+2}(n) + s^2 A_2^{p+2}(s; n) + B^{p+2}(n)\},$$

where the orders of  $A_1^{p+2}$  and  $A_2^{p+2}$  are less than  $1-\nu-\varepsilon$ . Here the highest order in  $B^{p+2}(n)$  is  $\sigma$  and it is given by the elements with the address  $(k_p+1, k_{p-1}, \dots, k_0, r)$  in

case of  $\ell > R_p - R_{p-1}$  and  $(k_p, k_{p-1}, \dots, k_0, r)$  in case of  $\ell \leq R_p - R_{p-1}$ .

By a suitable choice of B in the original operator P, we can take  $B^{p+2}$  such that it has only one non-zero element, (g,1)-element  $c_0 n^\sigma$  ( $c_0$  is a large constant), where  $g = k_p R_p + \sum_{j=0}^{p-1} (k_j - 1) R_j + r$  if  $\ell > R_p - R_{p-1}$  and  $g = \sum_{j=0}^p (k_j - 1) R_j + r$  if  $\ell \leq R_p - R_{p-1}$ .

We consider the characteristic polynomial of the full operator  $\hat{P}_{p+2}$ :

$$\det(\lambda I - \{n^{1-v-\epsilon} (\tilde{J}_{p+1}(s) + s K_{p+2}) + s A_1^{p+2}(n) + s^2 A_2^{p+2} + B^{p+2}(n)\}) \\ = \sum_{j=0}^m \alpha_j(s; n) \lambda^{m-j}.$$

$\alpha_g(s; n)$  has the form  $c_0 n^\delta s^\mu (1 + o(1))$ ,  $\delta = (g-1)(1-v-\epsilon) + \sigma$  and  $\exists \mu \in \mathbb{Z}_+$  ( $v = \sum_{j=0}^p \frac{\sigma}{R_{j-1} R_j}$ ). Here, we cannot find Jordan chains which are composed the vector of type  $s^\mu v$ ,  $v$ : constant vector.

By virtue of Proposition 1.1,  $\tilde{J}_{p+1} + K_{p+2}$  is nilpotent. Let us take  $N(s)$  which transforms  $\tilde{J}_{p+1} + s K_{p+2}$  to Jordan's normal form and set

$$(3.2) \quad \tilde{P}_{p+2} \equiv N^{-1} \cdot \hat{P}_{p+2} \cdot N = n^\sigma D_s - n^{1-v-\epsilon} J_{p+2} - C(s; n).$$

Here, the commutator  $n^\sigma N^{-1}(s) D_s N(s)$  has the same order  $\sigma$  as  $B^{p+2}(n)$  and it can give an influence on  $\alpha_g(s; n)$ . That is, setting

$$\det(\lambda I - n^{1-v-\epsilon} J_{p+2} - C(s; n)) = \sum_{j=0}^m \alpha'_j(s; n) \lambda^{m-j},$$

$\alpha'_g(s; n)$  may have the form  $(c_0 + c'_0(s)) n^\delta (1 + o(1))$ . However,  $c'_0(s)$  is decided by the principal part part of the original operator P and independent of  $B^{p+2}(n)$ . Thus,  $\alpha'_g(s, n) \neq 0$  and it has the order  $\sigma$ , if we take  $c_0$  sufficiently large.

Let  $\sigma$  be  $i_1/i_0$  ( $i_0, i_i \in \mathbb{N}$ ).  $1-v-\epsilon$  is also expressed as  $i_2/i_0$  ( $i_2 \in \mathbb{N}$ ). If  $1-v-\epsilon > \sigma$ , we can find a matrix  $N' \sim I + \sum_{h \in \mathbb{N}} n^{h/i_0} N_h(s)$  such that

$$(3.3) \quad Q \equiv N'^{-1} \circ \tilde{P}_{p+2} \circ N' = n^\sigma D_s - n^{1-v-\epsilon} J_{p+2} - C'(s; n),$$

where  $C'(s; n) = (C'(k, h, k', h')(s; n))$ ; block decomposition w.r.t.  $J_{p+2}$ ,

$$C'(k, h, k', h') = \begin{pmatrix} \gamma_1^{khk'h'} & 0 \\ \cdot & 0 \\ \gamma_k^{khk'h'} & 0 \end{pmatrix} \quad \text{and}$$

$$\gamma_j^{khk'h'} = \sum_{\substack{i \in \mathbb{Z} \\ i/i_0 + j(v+\varepsilon) < 1}} n^{i/i_0 + (j-1)(v+\varepsilon)} \gamma_{ji}^{khk'h'}(s).$$

(See, for example, V.M. Petkov [18] or rather its proof).

We say that a matrix which has the form as  $C'$  is admissible to  $n^{1-v-\varepsilon} J$ . By this transformation, the principal part of  $\alpha'_g(s; n)$  is preserved. From now on, we assume that  $1-v-\varepsilon > \sigma$ .

We introduce a notion :

**Definition 3.1.** (Stable coefficient of characteristic polynomial)

Let  $\tilde{C}$  be admissible to  $n^v J$ . We set

$$\det(\lambda I - n^v J - \tilde{C}(s; n)) = \sum_{j=0}^m \tilde{\alpha}_j(s; n) \lambda^{m-j}.$$

When the principal part of  $\tilde{\alpha}_j$  is preserved by any perturbation of order at most  $\sigma$ , we say that  $\alpha_j$  is a stable coefficient of the characteristic polynomial of full operator.

On the stable coefficients, the following proposition was obtained by W. Matsumoto [9].

**Proposition 3.1.** *If the original Cauchy problem is uniformly well-posed, the characteristic polynomial of full operator has no stable coefficient.*

We transform  $Q$  by shearing operator  $W'$  corresponding to  $J$  with weight  $\varepsilon_0 > 0$ ,  $\varepsilon_0$  : very small.

$$(3.4) \quad Q' \equiv W'^{-1} Q W' = n^\sigma D_s - n^{1-v-\varepsilon-\varepsilon_0} J - C''(s; n),$$

where the order of  $C''(s; n)$  is less than  $1-v-\varepsilon-\varepsilon_0$  and  $C''(s, n)$  is admissible to  $n^{1-v-\varepsilon-\varepsilon_0} J$ . Further, the elements which concern  $\alpha'_g(s; n)$  has the order  $\sigma + (g-1)\varepsilon_0$  in  $C''(s, n)$ . This implies that  $\alpha'_g(s; n)$  is stable in the characteristic polynomial of the full operator  $Q'$ , if we can find  $\sigma$  such that  $1-v-\varepsilon > \sigma > \frac{1}{2}$ . Then, when we can find a  $\sigma$  such that  $1-v-\varepsilon > \sigma > \frac{1}{2}$ , we arrive at a contradiction. Here,

the existence of such  $\sigma$  is equivalent to " $g \geq 3$ " and further equivalent to " $r \geq 2$  and if  $r = 2$ ,

$$\text{in } A_1 = (A_1(k, h, k', h'))_{\substack{1 \leq k, k' \leq 2 \\ 1 \leq h \leq m_k \\ 1 \leq h' \leq m_{k'}}} \text{ in (1.4) } \left( A_1(2, h, 2, h') = \begin{pmatrix} * & * \\ \alpha(h, h') & * \end{pmatrix} \right),$$

$(\alpha_{hh'})_{1 \leq h, h' \leq m_2}$  vanishes". " $r \leq 2$ " is equivalent to condition (1.5) with  $k = 0$  (Corollary

2) and the rest is equivalent to (1.5) with  $k = 1$ .

Q.E.D.

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