The Functional Calculus for the Laplacian on Lipschitz Domains

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Let $\Omega \subset \mathbb{R}^n$ be bounded, open, and connected. We define $W^{k,p}(\Omega)$ as the closure of $C^\infty(\overline{\Omega})$ in the norm $\|f\|_{k,p} = \left[\int_\Omega \sum_{|\alpha|\le k} |\frac{\partial^\alpha}{\partial x^\alpha} f(x)|^p \, dx\right]^{1/p}$. $W_0^{k,p}(\Omega)$ is the closure of $C_0^\infty(\Omega)$ in the same norm. The Dirichlet problem

$$(D) \begin{cases} -\Delta u = f & \text{in } \Omega \\ u|_{\partial\Omega} = 0 & \text{on } \partial\Omega \end{cases}$$

defines a positive selfadjoint operator on $L^2(\Omega)$. Thus there are eigenvalues $0 < \lambda_0 < \lambda_1 \le \lambda_2 \le \dots$ and a complete orthonormal sequence of eigenfunctions $\phi_0, \phi_1, \dots$ satisfying $-\Delta\phi_k = \lambda_k\phi_k$ in Ω $\phi_k \in C^\infty(\Omega)$ and the boundary condition $\phi_k \in W_0^{1,2}(\Omega)$. We denote this operator by $-\Delta_D$. Similarly, we consider the Neumann problem

$$(N) \begin{cases} -\Delta u = f & \text{in } \Omega \\ \frac{\partial u}{\partial \nu}|_{\partial\Omega} = 0 & \text{on } \partial\Omega . \end{cases}$$

The Poincaré inequality

$$\int_\Omega |f(x) - f_{ave}|^2 \, dx \le C \int_\Omega |\nabla f(x)|^2 \, dx$$

holds for all $f \in W^{1,2}(\Omega)$, under a mild regularity hypothesis on $\partial\Omega$. In this note will only be considering Lipschitz domains, for which it is easy to verify the Poincaré inequality. There are then eigenvalues $0 = \mu_0 < \mu_1 \le \mu_2 \le \dots$ and a complete orthonormal system of eigenfunctions $\psi_k \in W^{1,2}(\Omega) \cap C^\infty(\Omega)$ satisfying $-\Delta\psi_k = \mu_k\psi_k$ in Ω and verifying the boundary condition of (N) in the following weak sense:

$$\int_\Omega \nabla v \cdot \nabla \psi_k = \mu_k \int_\Omega v \psi_k \quad \text{for all } v \in W^{1,2}(\Omega) .$$

($\psi_0(x)$ is the constant function.) We denote the corresponding operator $-\Delta_N$.

One of the main objects of study in the functional calculus is the fractional powers of the operator. In this talk we will focus on the square root. Denote $A = (-\Delta D)^{1/2}$ and $B = (-\Delta_N)^{1/2}$.

If $f,g \in W_0^{1,2}(\Omega)$, then

$$\int_\Omega AfAg = \int_\Omega \nabla f \cdot \nabla g ,$$

so that $\|Af\|_{L^2(\Omega)} = \|\nabla f\|_{L^2(\Omega)}$. Similarly under the mild regularity hypothesis on $\partial\Omega$,

$$\int_\Omega BfBg = \int_\Omega \nabla f \cdot \nabla g$$

for all $f,g \in W^{1,2}(\Omega)$, so that $\|Bf\|_{L^2(\Omega)} = \|\nabla f\|_{L^2(\Omega)}$.

When $\partial\Omega$ is smooth, the theory of pseudodifferential operators and Calderón–Zygmund theory show that, in addition,

$$\|Af\|_{L^p(\Omega)} \simeq \|\nabla f\|_{L^p(\Omega)}$$

and

$$\|Bf\|_{L^p(\Omega)} \simeq \|\nabla f\|_{L^p(\Omega)}$$

for all p , $1 < p < \infty$. Here and elsewhere the notation $A \lesssim B$ means there is a constant C such that $A \leq CB$. $A \simeq B$ means both $A \lesssim B$ and $B \lesssim A$ hold.

<u>Theorem 1</u>. <u>Suppose that</u> $n \geq 3$ <u>and let</u> Ω <u>be a Lipschitz domain</u> (<u>i.e.</u>, $\partial\Omega$ <u>is given locally as the graph of a Lipschitz function</u>). <u>Then</u>

(a) <u>there is</u> $p_1 > 3$ <u>depending on the Lipschitz constant such that if</u> $1/p_0 + 1/p_1 = 1$,

$\|Af\|_{L^p(\Omega)} \lesssim \|\nabla f\|_{L^p(\Omega)}$ for $p_0 < p < \infty$

and $\|\nabla f\|_{L^p(\Omega)} \lesssim \|Af\|_{L^p(\Omega)}$ for $1 < p < p_1$.

In particular, $\|Af\|_{L^p(\Omega)} \simeq \|\nabla f\|_{L^p(\Omega)}$ for $p_0 < p < p_1$.

(b) The same result holds for the operator B.

(c) These inequalities are sharp. For instance, given $p > 3$, there is a Lipshitz domain Ω for which $\|\nabla f\|_{L^p(\Omega)}$ is not bounded by a constant times $\|Af\|_{L^p(\Omega)}$.

(d) If Ω is a C^1 domain, then (a) and (b) hold for all p, $1 < p < \infty$.

When $n = 2$, there is a similar result, but the range of exponents in parts (a), (b) and (c) is larger: $p_0 < p < \infty$ or $1 < p < p_1$, where $1/p_0 + 1/p_1 = 1$ and $p_1 > 4$, rather than $p_1 > 3$.

Let us give some applications of Theorem 1 to the inhomogeneous Dirichlet and Neumann problems and to the initial value problem for the heat equation. For purposes of comparison, we recall the estimates that hold when Ω has a C^∞ boundary. For u satisfying (D) or (N), we have

(α) $\|\nabla^2 u\|_{L^p(\Omega)} \le C\|f\|_{L^p(\Omega)}$ $\quad 1 < p < \infty$

(β) $\|\nabla u\|_{L^q(\Omega)} \le C\|f\|_{L^p(\Omega)}$ $\quad 1 < p < n,\ 1/q = 1/p - 1/n$

(γ) $\|u\|_{L^q(\Omega)} \le C\|f\|_{L^p(\Omega)}$ $\quad 1 < p < n/2,\ 1/q = 1/p - 2/n$.

The extent to which these inequalities may be extended to Lipshitz domains was treated by B. Dahlberg [1] in the case of the Dirichlet problem. He showed

(i) there exists a Lipschitz domain and $f \in C_0^\infty(\Omega)$ such that $\nabla^2 u \notin L^p(\Omega)$ for every p, $1 < p < \infty$. (Thus (α) fails.)

(ii) (β) holds for $1 < q < 3 + \epsilon$, where $\epsilon > 0$ depends on the Lipschitz contsnt, and this is sharp.

(iii) In the case of C^1 domains (β) holds for all q, $1 < q < \infty$.

Notice that in the case of the Dirichlet problem part (γ) obviously extends to arbitrary domains Ω. In fact, by the maximum principle, Green's function for the Dirichlet problem is smaller than the Newtonian potential on $\mathbb{R}^n$. The estimate on Ω then follows from the well–known fractional integral estimates on $\mathbb{R}^n$.

The first corollary of Theorem 1 is that we recover Dahlberg's results (ii) and (iii) and, in addition, the same results hold in the Neumann problem. In fact, $(-\Delta_D)^{-1} = A^{-2}$, so $\|\nabla u\|_q = \|\nabla A^{-2} f\|_q \lesssim \|AA^{-2} f\|_q = \|A^{-1} f\|_q \lesssim \|\nabla A^{-1} f\|_p \lesssim \|AA^{-1} f\|_p = \|f\|_p$. The first and last inequalities follow if $1 < q < p_1$ and $1 < p < p_1$. The middle inequality is just the usual fractional integral inequality.

We have also obtained a sharper counterexample than (i).

<u>Proposition</u>. <u>There is a</u> C^1 <u>domain and a solution</u> u <u>to</u> (D) <u>with</u> $f \in C_0^\infty(\Omega)$ <u>but</u> $\nabla^2 u$ <u>does not belong to</u> $L^1(\Omega)$. <u>There is a similar counterexample for the Neumann problem</u> (N).

This counter example can be expressed in terms of Green's function by saying that $\int_\Omega \nabla_x^2 G(x,y) f(y)\,dy$ does not belong to $L^1(\Omega)$. While the question in part (α) has a negative answer, a variant of this question posed by J. Nečas has a positive answer. He asked whether one can solve

$$(*) \qquad \begin{cases} -\Delta u = \operatorname{div} \vec{f} & \text{in } \Omega \\ u|_{\partial\Omega} = 0 & \text{in } \partial\Omega \end{cases}$$

for vectors $\vec{f}$ with components in $L^p(\Omega)$, with the natural estimate $\|\nabla u\|_{L^p(\Omega)} \le C\|\vec{f}\|_{L^p(\Omega)}$. In other words, what are the boundedness properties of $\int_\Omega \nabla_x \nabla_y G(x,y) \vec{f}(y)dy$? One can already see the difference between putting both derivatives on the x variable and splitting them between x and y from the fact that the case $p = 2$ is immediate.

Theorem 2. Let p_0 and p_1 be as in Theorem 1.

(a) Let u be a solution to (*). $\|\nabla u\|_{L^p(\Omega)} \le C\|\vec{f}\|_{L^p(\Omega)}$ for $p_0 < p < p_1$.

(b) The range of exponents p is best possible ($n \ge 3$).

(c) For C^1 domain the estimate holds for all p, $1 < p < \infty$.

(d) There are similar results for Neumann boundary data.

We can deduce Theorem 2(a) from Theorem 1(a) as follows. Rephrase Theorem 2(a) as

$$\|\nabla A^{-2} f\|_{L^p(\Omega)} \le C\|f\|_{W^{-1,p}(\Omega)}$$

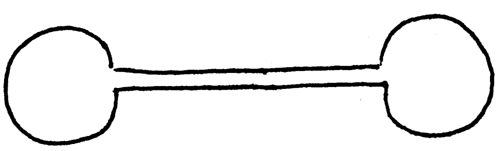
where $W^{-1,p}(\Omega)$ is the dual space of $W_0^{1,p'}(\Omega)$ with $1/p + 1/p' = 1$ or

$(a')_p \qquad \|A^{-2}f\|_{W_0^{1,p}(\Omega)} \leq C\|f\|_{W^{-1,p}(\Omega)}$.

For $p_0 < p < p_1$, Theorem 1(a) implies $\|Ag\|_{L^p(\Omega)} \simeq \|g\|_{W_0^{1,p}(\Omega)}$ and by duality $\|A^{-1}f\|_{L^p(\Omega)} \simeq \|f\|_{W^{-1,p}(\Omega)}$. But then $\|A^{-2}f\|_{W_0^{1,p}(\Omega)} \simeq \|AA^{-2}f\|_{L^p(\Omega)} = \|A^{-1}f\|_{L^p(\Omega)} \simeq \|f\|_{W^{-1,p}(\Omega)}$ as desired.

Since A is selfadjoint $(a')_p$ holds if and only if $(a')_{p'}$ holds. Thus for the counterexample in Theorem 2(b) it is enough to construct a Lipschitz domain for each $p > 3$ and solution u such that ∇u does not belong to $L^p(\Omega)$.

Let Γ be a circular cone of angle α. There exists function u harmonic in the complement ${}^c\Gamma$ given in polar coordinates by $u(r,\theta) = r^\lambda \phi(\theta)$ for $\lambda > 0$ depending on α and satisfying $u|_{\partial\Gamma} = 0$. When $n = 3$, it is easy to check that λ tends to zero as the operator α tends to zero. Let $\Omega = B \cap {}^c\Gamma$ where B is a ball centered at the vertex of Γ. Let $\psi \in C_0^\infty(B)$ be such that ψ is identically 1 in a neighborhood of the vertex. Then $v = u\psi$ satisfies $\Delta v \in C^\infty(\overline{\Omega})$, $v/\partial\Omega = 0$. However, $\nabla v \simeq r^{\lambda-1}$ which belongs to $L^p(\Omega)$ for $p < 3/(1-\lambda)$ only. For a counterexample in the Neumann problem we need to construct a cone for which there is a solution of the form $r^\lambda\phi(\theta)$ with $\lambda > 0$ arbitrarily small. Equivalently, we need to find a region in the sphere and an eigenfunction ϕ in the Neumann problem for the spherical Laplacian with eigenvalue arbitrarily close to zero. This can be accomplished with a region on the sphere with the shape



in which the width of the narrow strip joining the two disks tends to zero. The counterexample for $n = 3$ leads immediately to counterexample for $n > 3$ by adding extra independent variables.

Another application of Theorem 1 is to the heat equation in $\mathbb{R}_+ \times \Omega$ with initial data in $W_0^{1,p}(\Omega)$. Let $u(x,t) = (e^{-tA^2} f)(x)$ then $\partial_t u - \Delta u = 0$ in $\mathbb{R}_+ \times \Omega$, $u(x,t) = 0$ for $x \in \partial\Omega$ and $u(x,0) = f(x)$. $\|\nabla u(\cdot,t)\|_p \lesssim \|Au(\cdot,t)\|_p = \|e^{-tA^2} Af\|_{L^p(\Omega)} \lesssim \|Af\|_{L^p(\Omega)} \lesssim \|\nabla f\|_{L^p(\Omega)}$. The first inequality holds for $p < p_1$, the last for $p > p_0$. The middle inequality follows from the properties of the heat semigroup. (see [3].) Note that the estimate is independent of t. There is a similar result for the Neumann problem.

Finally, let us indicate some of the main ideas of the proof of Theorem 1 and the main obstacles. We will focus on the inequality

$$(*) \qquad \|A^{-1}f\|_{W_0^{1,p}(\Omega)} \le C\|f\|_{L^p(\Omega)} \qquad 1 < p < p_1$$

which is the same as $\|\nabla g\|_p \le C\|Ag\|_p$. We proceed by complex interpolation. First of all,

$$(1) \qquad \|A^{i\gamma}f\|_r \simeq \|f\|_r \qquad 1 < r < \infty$$

holds for any domain because of Stein's (Littlewood–Paley) multiplier theory for semigroups [3]. If we knew

$$(2) \qquad \|A^{-3/2}f\|_{W_0^{3/2,2}(\Omega)} \le C\|f\|_{L^2(\Omega)}$$

then we could deduce (*) for $3/2 < p < 3$ by interpolation. (The spaces $W^{\alpha,p}$ for fractional α are defined from their integer counterparts using complex interpolation.)

$\|A^{1/2}f\|_{W^{-1/2,2}(\Omega)} \simeq \|f\|_{L^2(\Omega)}$. If we substitute $f = A^{-1/2}g$ into (2) we find (2) is equivalent to

$$(2') \qquad \|A^{-2}g\|_{W_0^{3/2,2}(\Omega)} \le C\|g\|_{W^{-1/2,2}(\Omega)} .$$

Unfortunately this is false. Let us explain why. If (2′) were true, then we could solve $\Delta u = g$ in Ω with $u \in W_0^{3/2,2}(\Omega)$ for every $g \in W^{-1/2,2}(\mathbb{R}^n)$. Let $w \in W^{3/2,2}(\mathbb{R}^n)$. Then $g = \Delta w \in W^{-1/2,2}(\mathbb{R}^n)$, and there is a solution to $\Delta u = \Delta w$ in Ω satisfying $u \in W_0^{3/2,2}(\Omega)$. Let $v = w-u$, then v is harmonic in Ω and $v \in W^{3/2,2}(\Omega)$. We showed in [2] that for harmonic functions v in a Lipschitz domain $v \in W^{3/2,2}(\Omega)$ if and only if the restriction of v to $\partial\Omega$ belongs to $W^{1,2}(\partial\Omega)$. Since u vanishes on $\partial\Omega$, we can conclude that w restricted to $\partial\Omega$ belongs to $W^{1,2}(\partial\Omega)$. However, the restriction property $w \in W^{3/2,2}(\mathbb{R}^n)$ restricts to $W^{1,2}(\partial\Omega)$ does not hold for Lipschitz domains, so (2) and (2′) are false. We are indebted to Guy David for pointing out that this restriction property fails.

Not only does the endpoint estimate fail, but estimates in Theorem 1(a) for $p_1 > p > 3$ require results of type (2) for exponents $p > 2$. The correct estimate is

$$(3) \qquad \|A^{-2}f\|_{W^{1+\frac{1}{p}-\epsilon,p}(\Omega)} \le C\|f\|_{W^{-1-\epsilon+\frac{1}{p},p}(\Omega)}$$

for $0 < \epsilon < 2-2/p+\delta'$, $1 < p < 2 + \delta$, δ , δ' , ϵ sufficiently small and positive. Notice that we are able to treat exponents $p = 2$ and slightly larger only at the expense of decreasing the order of smoothness. This corresponds to a restriction lemma: Functions in $W^{1+\frac{1}{p}-\epsilon,p}(\Omega)$ restrict to the space $\Lambda_{1-\epsilon}^{p,p}(\partial\Omega)$ for $\epsilon > 0$, where $\Lambda_{1-\epsilon}^{p,p}(\partial\Omega)$ is the ~~space~~ Besov space $\left\{ f \in L^p(\partial D) : \iint_{\partial D\, \partial D} \frac{|f(x)-f(y)|^p}{|x-y|^{n-1+(1-\epsilon)p}}\, d\sigma(x)\, d\sigma(y) < \infty \right\}$

The main point of the proof is that there is also converse to the restriction lemma:

if $\Delta v = 0$ in Ω and $v|_{\partial\Omega}$ belongs to $\Lambda^{p,p}_{1-\epsilon}(\partial\Omega)$, then v belongs to $W^{1+\frac{1}{p}-\epsilon,p}(\Omega)$, $1 < p < 2 + \delta$.

References

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