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# Resolution blowups, spectral convergence and quasi-asymptotically conical spaces

Rafe Mazzeo

## 1. Introduction

In the class of all stratified spaces are those spaces obtained by an iterated coning procedure. We define this subclass of ‘iterated cone-edge’ spaces in more detail below. A complete analysis of the natural elliptic operators in this class of spaces remains a challenge, although by now there are many approaches and proposals for how to proceed. In the geometric context this was initiated by Cheeger [2], but the ideas proposed by Melrose [12] will likely provide the best framework for carrying this out in detail; see also [14] and [13] for other approaches. For manifolds with isolated conic or simple edge singularities, the theory is quite refined, see [11], [9], [6] and references therein.

The present paper is concerned not only with this class  $\mathcal{I}$  of singular spaces, but also with two related classes of smooth manifolds: the class  $\mathcal{D}$  of ‘resolution blowups’ of elements of  $\mathcal{I}$ , and the class  $\mathcal{Q}$  of quasi-asymptotically conic (QAC) spaces, which (by definition) have ‘link’ an element of  $\mathcal{D}$ . One of the main goals of this paper is to describe more precisely the web of connections between  $\mathcal{I}$ ,  $\mathcal{D}$  and  $\mathcal{Q}$ , and to explain how the analysis of elliptic operators on an element in any one of these three classes relies on the analysis of induced operators on ‘subsystem spaces’ in the other two classes. We state some basic results in the elliptic theory on each type of space, and in particular sketch the proof of a result about the convergence of the spectrum for generalized Laplacians on resolution blowups as the space collapses to one with iterated cone-edge singularities. The proof of this last result is not self-contained since it is part of a larger induction scheme which also involves QAC spaces. It is a generalization of the proof in the ‘depth 1 case’, which appears in the recent work of Rowlett [15]. Complete details concerning the geometry of these various classes of spaces, as well as a thorough treatment of this spectral convergence result and various results concerning Fredholm properties of generalized Laplacians on weighted Sobolev and Hölder spaces, will appear in a forthcoming joint work with Anda Degeratu [4].

Before proceeding, let us state that in this paper a generalized Laplacian will refer to an elliptic operator of the form  $L = \nabla^* \nabla + R$ , acting on sections of some

bundle  $E$  over a (possibly singular) Riemannian manifold  $X$ . We assume that  $E$  is a parallel subbundle of the tensor bundle of  $X$  (possibly tensored with the spinor bundle and an auxiliary flat bundle), and that  $\nabla$  is the Levi-Civita connection. The self-adjoint endomorphism  $R$  of  $E$  is assumed to be constructed using the curvature tensor and its covariant derivatives, and is conformally covariant of degree  $-2$  with respect to homothetic rescalings of the metric. A useful feature of these operators is that they are canonically associated to a metric, and thus can be transferred from one space to another without having to list a host of technical assumptions.

Before stating the main result about spectral convergence, it is necessary to give a rather lengthy discussion of the geometric setting, and of the precise modes of geometric degeneration treated here. In broad terms, we are interested in families of metrics  $h_\epsilon$  on a compact smooth manifold  $Y$  such that, as  $\epsilon \searrow 0$ ,  $(Y, h_\epsilon)$  converges to a Riemannian stratified space  $(Y_0, h_0)$  in the class  $\mathcal{I}$  of iterated cone-edge spaces; in this singular limit, some finite collection of QAC spaces is ‘pinched off’. The spectral and Fredholm properties of the induced operators  $L_Z$  on each of these QAC spaces plays a role in understanding whether the spectrum of  $L_{Y, h_\epsilon}$  converges to that of  $L_{Y_0, h_0}$ .

Let us begin with the simplest example. Suppose that  $(Z, g_Z)$  is an asymptotically conical (AC) manifold. This means that there exists some compact subset  $K = K_Z \subset Z$  such that  $Z \setminus K_Z$  is diffeomorphic to a product  $[1, \infty) \times \Sigma$ . Here  $\Sigma$  is a smooth compact manifold endowed with a Riemannian metric  $\kappa$ , and we have specified a diffeomorphism identifying  $\partial K_Z$  with  $\{1\} \times \Sigma$ . The coordinate  $\rho \in [1, \infty)$  is called a radial function, and in terms of this data, the metric  $g_Z$  takes the form

$$g_Z \sim d\rho^2 + \rho^2 \kappa$$

on this neighbourhood of infinity. Here and elsewhere in this paper, we shall use the shorthand notation that a metric  $g$  is similar to ( $\sim$ ) some normal form to mean that  $g$  is equal to this normal form plus an error term which decays in the relevant asymptotic region (here, as  $\rho \rightarrow \infty$ ). The manifold  $\Sigma$  is called the link of  $Z$ . The homothetic rescalings  $(Z, \epsilon^2 g_Z)$  converge in the pointed Gromov-Hausdorff sense to the complete Riemannian cone  $(C(\Sigma), d\hat{\rho}^2 + \hat{\rho}^2 \kappa)$ ; here, as usual,  $C(\Sigma)$  is the product  $[0, \infty)_{\hat{\rho}} \times \Sigma$  where the boundary  $\{0\} \times \Sigma$  is collapsed to a point (as indicated by the degeneration of the metric at  $\hat{\rho} = 0$ ). In this collapse, all the topology of  $K_Z$  is squeezed to a point. Conversely, we can also regard the space  $Z$  as a resolution of this conic singular space  $C(\Sigma)$ .

Suppose now that  $(Y_0, h_0)$  is a compact Riemannian space with isolated conic singularity, with link  $(\Sigma, \kappa)$ , at a point  $p$ . Thus, near  $p$ ,  $h_0 \sim dr^2 + r^2 \kappa$  as  $r \searrow 0$ . We may now ‘resolve’ the singularity of  $(Y_0, h_0)$  using the AC space  $(Z, g_Z)$  above, since both have the same link  $(\Sigma, \kappa)$ , to obtain a smooth compact manifold  $Y$  and a family of Riemannian metrics  $h_\epsilon$  such that  $(Y, h_\epsilon) \rightarrow (Y_0, h_0)$  not only in the sense of Gromov-Hausdorff limits, but also in  $\mathcal{C}^\infty$  away from the conic point. More precisely,  $Y$  is obtained by replacing a small neighbourhood of the conic point in  $Y_0$  with some compact truncation of  $Z$ . The metric  $h_\epsilon$  is defined by pasting together  $h_0$  and  $\epsilon^2 g_Z$ . This is the first example of what we shall call a resolution blowup (here, of  $Y_0$  along  $Z$ ). One could also let  $(Y_0, h_0)$  have a simple edge singularity, i.e. a single singular stratum  $S$ , some collar neighbourhood of which is modelled on a cone bundle over  $S$ , with each fibre a truncated cone over  $\Sigma$ . One can then carry out this resolution

process fibrewise to obtain a smooth manifold and a degenerating family of metrics, much as before.

A central question, and the main one we consider here, is whether the spectrum of  $L_{Y,h_\epsilon}$  converges to that of  $L_{Y_0,h_0}$ . Note that since  $Y_0$  is incomplete, we must specify a self-adjoint extension of this operator and (subject to some mild restrictions on the curvature term  $R$ ) we shall always use the Friedrichs extension. It turns out that in order to answer this, one needs to know something about the mapping properties for the operator  $L_Z$  on weighted Sobolev spaces. In other words, this spectral convergence problem relies on Fredholm theory of the elliptic operator  $L_Z$  on the complete space  $Z$ .

This Fredholm theory on  $Z$  is an interesting problem in its own right. It is important here that the term of order 0 in  $L_Z$  scales like  $\rho^{-2}$ . (The analysis of operators with more general nondecaying lower order terms, for example,  $L_Z - \lambda$  with  $\lambda > 0$ , turns out to have a very different nature.) For simplicity, we just recall the ‘classical’ result for a perturbation  $\Delta + V$  of the scalar Laplacian for  $g_Z$ , where we assume that  $V \in C^\infty(Z)$ ,  $V \sim \alpha/\rho^2$ ,  $\alpha \in \mathbb{R}$ , as  $\rho \rightarrow \infty$  (where the Laplacian is a nonnegative operator). Then

$$\Delta_{g_Z} + V : \rho^\delta L^2(M; dV_{g_Z}) \longrightarrow \rho^{\delta-2} L^2(M; dV_{g_Z})$$

is Fredholm provided  $\delta$  lies in a range of weights  $\mathbb{R} \setminus \{\pm\gamma_j\}$ , where the numbers  $\gamma_j$  are determined by the spectrum  $\{\lambda_j + \alpha\}$  of the induced operator  $\Delta_k + \alpha$  on the link  $\Sigma$ :  $\gamma_j = \sqrt{\frac{(n-2)^2}{4} + \lambda_j + \alpha}$ . This sets the stage for a closer study of the regularity properties of solutions, Hodge and index theory, etc.

We now explain an interesting generalization of AC geometry and of this Fredholm result. There is a class of smooth manifolds in complex algebraic geometry, called QALE (for quasi-asymptotically locally Euclidean) spaces. These are obtained as so-called crepant resolutions of complex orbifolds  $\mathbb{C}^n/\Gamma$ , where  $\Gamma \subset \mathrm{SU}(n)$  is a finite subgroup. If  $\Gamma$  acts freely away from 0, the corresponding QALE space is asymptotically conical in the sense above; however, in general, the fixed point set of  $\Gamma$  is a union of subspaces  $\{W_j\}$ , each stabilized by a different isotropy subgroup of  $\Gamma$ . These crepant resolutions,  $Z \rightarrow \mathbb{C}^n/\Gamma$  are of interest for many reasons, but particularly because, as proved by Dominic Joyce, they all admit complete Ricci-flat Kähler-Einstein metrics. We refer to [7] for a description of (and more references on) these resolutions as well as Joyce’s construction of these metrics using the complex Monge-Ampere equation. QALE spaces  $(Z, g_Z)$  have many interesting properties, but we emphasize only a few of these here. First, if  $Z$  is a resolution of the quotient  $\mathbb{C}^n/\Gamma$ , then the homothetic rescalings  $(Z, \epsilon^2 g_Z)$  converge to  $\mathbb{C}^n/\Gamma$  in pointed Gromov-Hausdorff norm as  $\epsilon \rightarrow 0$ . There is a smoothly bounded compact submanifold  $K_Z$ , the exterior of which is diffeomorphic to  $[1, \infty) \times Y$ , as in the AC case, with radial function  $\rho$  as the first coordinate; in terms of this diffeomorphism,  $g_Z \sim d\rho^2 + \rho^2 h(\rho)$  where  $\{h(\rho)\}$  is a family of metrics on the link  $Y$  such that as  $\rho \rightarrow \infty$ ,  $(Y, h(\rho))$  converges to  $S^{2n-1}/\Gamma$  (with its round orbifold metric), which is an element of  $\mathcal{I}$ . This is a geometric correspondence dual to the one above; namely, the rescaled cross-sections of QALE spaces collapse onto orbifolds, and the geometry of this collapse is precisely the type in which we are interested. Joyce obtained certain weaker mapping properties of the scalar Laplacian (which was all he needed) using the maximum principle, but in order to obtain the Fredholm theory for systems, e.g. for the generalized Laplacian  $L_Z$ , with an optimal range of weights, one needs

to understand the spectral convergence for the family of induced operators on this degenerating link. This is a problem of lower geometric complexity (i.e. lower depth) than  $Z$  itself, so the spectral convergence of  $L_{Y,h(\rho)}$  can be analyzed in terms of the Fredholm theory of still simpler QALE spaces.

The general scheme of what we are proposing to do should now be clear. We describe this in a slightly broader setting which leaves the confines of the algebro-geometric setting of the original QALE construction, and recasts the resolution process more geometrically.

Define  $\mathcal{I}$  to be the class of compact stratified spaces with iterated cone-edge singularities and with a compatible Riemannian metric;  $\mathcal{I}_k$  denotes the subset consisting of spaces which can be defined using at most  $k$  iterated conings. Next,  $\mathcal{D}_k$  is the class of compact spaces  $Y$  equipped with a ‘blowdown map’  $\beta : Y \rightarrow Y_0$ ,  $Y_0 \in \mathcal{I}_k$ , and a family of Riemannian metrics  $h_\epsilon$  so that  $(Y, h_\epsilon) \rightarrow (Y_0, h_0)$  in the Gromov-Hausdorff sense, and in  $\mathcal{C}^\infty$  away from the singular locus of  $Y_0$ ; we write  $\mathcal{D} = \cup_k \mathcal{D}_k$ . Finally,  $\mathcal{Q}$  is the class of quasi-asymptotically conical (QAC) manifolds, again filtered by ‘depth’, so  $\mathcal{Q} = \cup_k \mathcal{Q}_k$ . Each  $(Z, g_Z) \in \mathcal{Q}_k$  is a smooth complete Riemannian manifold with the following properties; there exists a compact smoothly bounded submanifold  $K_Z$  such that  $Z \setminus K_Z$  is diffeomorphic to  $[1, \infty)_\rho \times Y$  and with respect to this diffeomorphism,  $g_Z \sim d\rho^2 + \rho^2 h_{1/\rho}$ , where  $(Y, h_\epsilon) \in \mathcal{D}_k$ .

Each resolution  $Y \rightarrow Y_0$ , with  $Y \in \mathcal{D}_k$ ,  $Y_0 \in \mathcal{I}_k$ , is defined by an iterative process of attaching collections of spaces  $\{Z_j \in \mathcal{Q}_j\}_{j < k}$ . In the example above, the AC space  $(Z, g_Z)$  and its link  $(\Sigma, \kappa)$  lie in  $\mathcal{Q}_0$  and  $\mathcal{D}_0$ , respectively. The degenerating family  $(Y, h_\epsilon)$  we constructed by gluing  $(Z, g_Z)$  to the complement of the singularity in the conic space  $(Y_0, h_0) \in \mathcal{I}_1$  is an element of  $\mathcal{D}_1$ . Next, one can construct an element  $(Z', g_{Z'}) \in \mathcal{Q}_1$  with  $(Y, h_\epsilon)$  as its link. This requires a choice of a smooth compact manifold with boundary  $K_{Z'}$  such that  $\partial K_{Z'} = Y$ ; this is the new part of the resolution, and of course there is considerable freedom in choosing this (provided that it exists at all!). Outside of  $K_{Z'}$  we specify that  $Z'$  is diffeomorphic to  $[1, \infty)_{\rho'} \times Y$  with  $g_{Z'} \sim d(\rho')^2 + (\rho')^2 h_{1/\rho'}$ , as before, and hence  $(Z', \epsilon^2 g_{Z'}) \rightarrow (C(Y_0), d\hat{\rho}^2 + \hat{\rho}^2 h_0)$  (where  $\hat{\rho}$  is the limit of the variable  $\epsilon\rho'$ ). In this limit,  $K_{Z'}$  collapses to a point.

To summarize, there is a natural map  $\mathcal{D}_k \rightarrow \mathcal{I}_k$ ; this is not surjective since there are many spaces  $(Y_0, h_0) \in \mathcal{I}_k$  which admit no smoothings at all. There is also a natural map  $\mathcal{Q}_k \rightarrow \mathcal{D}_k$  obtained by taking the link family  $(Y, h_\epsilon)$  of a QAC space  $(Z, g_Z)$ , which is also not surjective; the range consists of those manifolds  $Y$  (with appropriate degenerating metric) which smoothly bound, but even on this subset there is no obvious inverse mapping (it is, of course, an interesting question whether there are minimal, or in some other way canonical, fillings in the smooth category). Finally, the resolution blowup is defined on some appropriate subset of  $\mathcal{I}_k \times \cup_{j < k} \mathcal{Q}_j$  and has image in  $\mathcal{D}_k$ .

There are numerous analytic problems to be studied in this setting. We mention just a few. One direction is to study generalized Laplace operators on QAC spaces; a first step is to prove that these operators are Fredholm on natural weighted spaces, and after that to go on to study the Hodge theory, index theory of Dirac-type operators and finally more subtle problems in scattering theory, etc. Another direction is to study these generalized Laplacians on the compact smooth spaces  $(Y, h_\epsilon) \in \mathcal{D}_k$ . Here the initial step is to prove convergence of  $\text{spec}(L_{Y,h_\epsilon})$  as  $\epsilon \rightarrow 0$ ; further refinements include the limiting behaviour of spectral invariants, such as the determinant, eta invariant, etc. These two directions are closely linked: the range of

allowable weights for the Fredholm theory of  $L_Z$ , where  $(Z, g_Z) \in \mathcal{Q}_k$ , is determined by the limiting behaviour of the spectrum of the induced generalized Laplacian on its link  $(Y, h_\epsilon) \in \mathcal{D}_k$ ; conversely, aspects of the Fredholm theory on the subsystem QAC spaces  $(Z_j, g_{Z_j})$  appearing in the resolution blowup  $Y \rightarrow Y_0$  are needed to understand the limiting behaviour of the spectrum on  $(Y, h_\epsilon)$ .

Concerning the history, Joyce [7] obtained the Fredholm theory for the scalar Laplacian on QALE spaces. His methods do not extend readily to systems, and are also not sharp with respect to the range of allowable weights. The Hodge theory for QALE spaces of depth 1, was studied recently by Carron [1], using a very interesting refinement of the Mayer-Vietoris sequence adapted to certain noncompact settings. As is clear in his proof, the underlying complex geometry of a QALE space allows for various simplifications of the computation, making it more tractable than is likely for a general QAC space. The index theory has, to my knowledge, not been attempted in this setting. There is a much more extensive history for the study of the spectrum of various types of Laplacians on collapsing families of compact Riemannian manifolds. On the one hand is the work of Cheeger-Colding [3], Lott [8] and Ding [5], which addresses much more general types of collapse. On the other hand, McDonald [10] studied the resolvent for a simple form of conic collapse, while the very recent work of Rowlett [15] specifically treats the spectral convergence and also obtains detailed results about the convergence of the heat kernel for  $(Y, h_\epsilon) \in \mathcal{D}_1$ .

The goals in this paper are, first to lay out this general program, and second, to sketch the proof of spectral convergence for generalized Laplacians on degenerating families  $(Y, h_\epsilon)$ . This is the technically easiest step in the whole chain, yet it gives some idea of the arguments used. Since this is part of an induction, this proof is unavoidably not self-contained, but relies at one point on some aspects of the mapping properties on QAC spaces of lower depth. More specifically, we prove the

**Theorem (Spectral Convergence).** *Let  $(Y, h_\epsilon) \in \mathcal{D}_k$ , and suppose that  $L_\epsilon = \nabla^* \nabla + R$  is a generalized Laplacian with respect to the metric  $h_\epsilon$  acting on sections of some subbundle of the full tensor bundle of  $Y$ , as described earlier, with  $R \geq 0$ . Denote by  $\{Z_j\}_{j < k}$  the collection of QAC spaces of depth  $j < k$  used in the resolution blowup  $Y \rightarrow Y_0$ . We assume that each  $L_{Z_j} \geq 0$ , and that moreover 0 is not an  $L^2$  eigenvalue of any  $L_{Z_j}$ , for all  $j < k$ . If  $\Lambda_\epsilon = \{\lambda_j(\epsilon)\}$  is the spectrum of  $L_\epsilon$  and  $\Lambda_0$  the spectrum of the Friedrichs extension of  $L_{Y_0, h_0}$ , then the set of accumulation points of  $\Lambda_\epsilon$  as  $\epsilon \rightarrow 0$  is equal to  $\Lambda_0$ , and in fact  $\Lambda_\epsilon \rightarrow \Lambda_0$  as sets with multiplicity.*

As already noted, the argument used to prove this theorem is a generalization of the one in [15], which treats the case  $k = 1$  (and achieves far more than just this spectral convergence), and that one in turn is based upon ideas developed in a larger ongoing project, joint with Anda Degeratu [4], to study the Fredholm theory and other aspects of the elliptic theory for such operators on general QAC spaces. This last-cited paper will contain a more detailed development of all of the definitions and basic results here, and for this reason the present discussion will be somewhat abbreviated.

The limitation to ‘geometric Laplacians’  $L = \nabla^* \nabla + R$  is not serious, of course, and there are analogous results for more general operators (including ones without diagonal principal part), and also for first-order (e.g. Dirac-type) operators.

Extending these results to such operators requires some laborious structure conditions, which follow directly from the metric asymptotics when discussing these Laplace-type operators.

Finally, it is worth re-emphasizing that this result about spectral convergence is one of the weaker results one can prove in this setting, as will be clear from the rather elementary nature of the proof. In particular, we obtain no information about uniformity of the rate of convergence, and hence this cannot be used directly to study spectral invariants, nor any of the more refined problems beyond the Fredholm theory on QAC spaces. It will probably be necessary to carry out a more sophisticated parametrix construction (as has been done for  $\mathcal{D}_1$  in [15]) to achieve these ends; such a construction will unfortunately be quite intricate. This has prompted the development of the more elementary arguments suggested here.

Thanks are due to Anda Degeratu for allowing me to report on our joint work about QAC spaces; she and also Pierre Albin read this paper carefully and made many valuable remarks. I have also had helpful discussions concerning generalized conic degeneration with Julie Rowlett. A detailed microlocal treatment of analysis on iterated cone-edge spaces was planned and sketched out a (very) long time ago with Richard Melrose, to whom I owe apologies for letting that project lie dormant for so long; his point of view on degeneration problems has, of course, been very influential on my own, though that may not be so evident here. Finally, thanks are due to Frank Pacard, with whom I have been discussing problems and methods closely related to the ones here for many years.

## 2. Resolution blowups

We now give a more careful description of the geometry of iterated cone-edge spaces, resolution blowups and QAC spaces. We shall always be working with Riemannian spaces, and so shall describe the differential topology and the metric simultaneously.

### 2.1. Iterated cone-edge spaces

Let us begin with the basic notion of a Riemannian cone. Suppose that  $(\Sigma, h)$  is a stratified Riemannian space. The (complete) cone over  $\Sigma$  is space  $[0, \infty)_r \times \Sigma / \sim$ , where the boundary  $\{0\} \times \Sigma$  is identified to a point, with metric  $dr^2 + r^2\kappa$ ; the truncated cone  $C_{a,b}(\Sigma)$  denotes the subset where  $a \leq r \leq b$ . Note that any singular stratum  $S \subset \Sigma$  induces a singular stratum  $C(S)$  of one higher dimension in the cone  $C(\Sigma)$ .

**Definition.** *For each  $k \geq 0$  the class of iterated cone-edge spaces of depth  $k$ , denoted  $\mathcal{I}_k$ , is defined by induction on  $k$ . An iterated cone-edge space of depth 0 is a compact smooth manifold. A stratified space  $X$  lies in  $\mathcal{I}_k$  if for any  $p \in X$ , if  $S$  is the open singular stratum containing  $p$  and  $\dim S = \ell$ , then there exists a neighbourhood  $\mathcal{U}$  of  $p$  in  $X$  which is diffeomorphic to the product  $\mathcal{V} \times C_{0,1}(\Sigma)$  where  $\mathcal{V} \subset \mathbb{R}^\ell$  is an open Euclidean ball diffeomorphic to a neighbourhood in  $S$  and  $\Sigma \in \mathcal{I}_j$  for some  $j < k$ , and the integer  $n = \ell + \dim C(\Sigma)$  is independent of the point  $p$ , and is called the dimension of  $X$ . If  $\dim S > 0$ , then we say that it is an edge in  $X$  with fibre  $C(\Sigma)$  and link  $\Sigma$ . Thus  $X \in \mathcal{I}_k$  if it can be formed by a  $k$ -fold iterated coning or edging procedure. An iterated cone-edge metric  $g$  on  $X$  is by definition one which respects*

this diffeomorphism, i.e. is locally of the form  $g \sim dr^2 + r^2h + \kappa$ , where  $h$  is an iterated cone-edge metric on  $\Sigma$  and  $\kappa$  is a metric on  $S$ . The entire class of iterated cone-edge spaces  $\mathcal{I}$  is the union over all  $k$  of these subclasses  $\mathcal{I}_k$ .

It is often helpful, and not too restrictive, to assume slightly more structure on the metric  $g$  near a nonisolated singular stratum; namely, suppose that the link of this stratum with metric  $h + \kappa$  is a Riemannian submersion over the stratum  $(S, \kappa)$ .

A useful tool in the study of iterated cone-edge spaces is the notion of radial blowup. This is a natural operation which replaces the singular space  $X \in \mathcal{I}_k$  with a manifold with corners  $\bar{X}$ . The radial blowup of a truncated cone  $C_{0,1}(\Sigma)$  is obtained by replacing the vertex by the boundary  $\{0\} \times \Sigma$ . (The conic metric lifts to be degenerate at this boundary.) The iterated radial blowup of a space  $X \in \mathcal{I}_k$  is defined by radially blowing up the strata in order of decreasing codimension.

**Proposition.** *If  $X \in \mathcal{I}_k$ , then its iterated radial blowup  $\bar{X}$  is a manifold with corners of codimension  $k$ . Every boundary face  $H \subset \bar{X}$  is the total space of a fibration, where the base is the iterated radial blowup  $\bar{S}$  of a stratum  $S \subset X$ , which is itself a manifold with corners, and with fibre the iterated radial blowup of the link  $\Sigma$  for that stratum. There is a natural blowdown map  $\pi_X : \bar{X} \rightarrow X$ .*

Let  $\bar{p} \in \bar{X}$  with  $p = \pi_X(\bar{p})$  lying on the interior of some stratum  $S$ , and fix a diffeomorphism of an neighbourhood of  $p$  in  $X$  with a product  $\mathcal{V} \times C_{0,1}(\Sigma)$ ,  $\Sigma \in \mathcal{I}_j$ . There is a coordinate system near  $\bar{p}$  comprised of radial coordinates  $r_0, \dots, r_\ell$  for some  $\ell \leq j$ , with each  $r_i \in [0, 1)$ , and coordinates in a product of Euclidean balls,  $(y, z) \in B^r \times B^s$ ,  $r + s = n - \ell - 1$ . Here  $r_0$  is the radial function for the cone  $C(\Sigma)$  and  $y$  is a local coordinate on  $S$ ; the remaining coordinates  $(r_1, \dots, r_\ell, z)$  are of the same inductively determined type on the radial blowup of the link  $\bar{\Sigma}$ .

We conclude with one final remark. If  $Y_0 \in \mathcal{I}_k$  and  $S_k$  is the stratum of greatest depth,  $k$ , then  $Y_0 \setminus S_k$  has iterated cone-edge singularities only up to depth  $k - 1$ . All of the geometric constructions here are essentially local, and since the inductive hypothesis will be that we have carried out our various constructions for all spaces of depth less than  $k$ , completing the inductive step will simply involve showing that we can extend these constructions over this final depth  $k$  stratum, or over the smooth spaces which converge to it.

## 2.2. QAC spaces and resolution blowups

We have already provided some indication of the definition of QAC manifolds, and of the way these can be used to resolve iterated cone-edge spaces. We now carry this out more precisely, and give two interlocking definitions, one for the class  $\mathcal{Q}$  of QAC spaces and the other for the class  $\mathcal{D}$  of resolution blowups of iterated cone-edge spaces. Both  $\mathcal{Q}$  and  $\mathcal{D}$  are filtered by subclasses  $\mathcal{Q}_k$  and  $\mathcal{D}_k$ , respectively;  $\mathcal{Q}_0$  consists precisely of the AC spaces, and  $\mathcal{D}_0$  consists of compact smooth manifolds with fixed metric.

Before embarking on the general case, we begin with a more careful description than was given in the introduction of the resolution blowup  $(Y, h_\epsilon) \in \mathcal{D}_1$  of  $Y_0 \in \mathcal{I}_1$ . Suppose that  $Y_0 \in \mathcal{I}_1$  has a simple edge singularity along the stratum  $S = S_1$ . Since it has greatest depth,  $S$  is compact and smooth, and by definition of  $\mathcal{I}_1$ , some neighbourhood of it in  $Y_0$  is diffeomorphic to a bundle over  $S$  with fibre a



truncated cone  $C_{0,1}(\Sigma)$ , where  $\Sigma \in \mathcal{D}_0$ . In this neighbourhood, the edge metric has the form  $h_0 \sim dr^2 + r^2 h_\Sigma + \kappa_S$ . Suppose that  $(Z, g_Z) \in \mathcal{Q}_0$  has link  $(\Sigma, h_\Sigma)$ ; thus there exists a smooth compact submanifold with boundary  $K = K_Z \subset Z$  such that  $Z \setminus K \cong C_{1,\infty}(\Sigma)$  and  $g_Z \sim d\rho^2 + \rho^2 h_\Sigma$ . The metrics  $g_0$  and  $g_Z$  on  $Z$  are compatible in the sense they induce the same metric  $h_\Sigma$  on the link.

For any  $a < 1$ , denote by  $B_a(S)$  the bundle over  $S$  with fibre the truncated cone  $C_{0,a}(\Sigma)$ . Similarly, for any  $b > 1$ , denote by  $Z_b$  the truncation  $K \cup \{\rho < b\} \subset Z$ , and set  $\tilde{B}_b(S)$  equal to the bundle over  $S$  where each cone  $C_{0,a}(\Sigma)$  is replaced by  $Z_b$ . Thus  $\tilde{B}_b(S)$  may be regarded as a resolution of  $B_a(S)$ . The resolution blowup of  $Y_0$  is now defined by excising the neighbourhood  $B_a(S)$  from  $Y_0$  and replacing it by  $\tilde{B}_{1/a}(S)$ . This can be done at any scale: thus, for any  $\epsilon < 1$ , define the gluing map

$$\psi_\epsilon : \tilde{B}(S)_{1/\epsilon} \setminus \overline{\tilde{B}(S)_1} \longrightarrow B_1(S) \setminus \overline{B_\epsilon(S)}, \quad \psi_\epsilon(\rho, y, s) = (\epsilon\rho, y, s), \quad y \in \Sigma, s \in S.$$

This allows us to define the smooth manifold

$$Y_\epsilon = Y_0 \setminus \overline{B_\epsilon(S)} \cup_{\psi_\epsilon} Z_{1/\epsilon}.$$

The metrics  $h_0$  and  $g_Z$  glue together in an obvious way on this space to produce a degenerating family of metrics  $h_\epsilon$ . Note that all the manifolds  $Y_\epsilon$ ,  $\epsilon > 0$ , are diffeomorphic to one another, but the metric  $h_\epsilon$  is most naturally defined via the attaching map  $\psi_\epsilon$ , so it is useful to keep the notation  $Y_\epsilon$  to remind us of the scale of the gluing map. In any case, we have now defined  $(Y_\epsilon, h_\epsilon)$ , which is an element of  $\mathcal{D}_1$ . Following this same convention, we shall often write an element of  $\mathcal{D}_k$  as  $(Y_\epsilon, h_\epsilon)$  even though the manifolds  $Y_\epsilon$  are mutually diffeomorphic when  $\epsilon > 0$ .

Following this gluing scheme we now proceed to define the higher depth elements of each of these classes by induction on  $k$ . Note that we have already defined  $\mathcal{I}_k$  for every  $k$ .

### 2.2.1. $\mathcal{D}_k \rightsquigarrow \mathcal{Q}_k$

Let  $(Y_\epsilon, h_\epsilon) \in \mathcal{D}_k$ , and suppose that  $Y = \partial K$  for some compact smooth manifold  $K$ . We then define  $Z \in \mathcal{Q}_k$  as the union of  $K$  with  $[1, \infty) \times Y$ , where  $\partial K$  is identified with  $\{1\} \times Y$ , and endow it with the metric

$$g_Z \sim d\rho^2 + \rho^2 h_{1/\rho},$$

at least on  $Y \times (1, \infty)$ ; its extension over  $K$  is arbitrary. The space  $(Z, g_Z)$  is, by definition, an element of  $\mathcal{Q}_k$  associated to the degenerating family  $(Y_\epsilon, h_\epsilon)$ .

Conversely, and again by definition, the link of any  $(Z, g_Z) \in \mathcal{Q}_k$  is an element  $(Y_\epsilon, h_\epsilon) \in \mathcal{D}_k$ .

Slightly more generally, if  $W_\epsilon$  is the total space of a fibration with fibre  $Y \cong Y_\epsilon \in \mathcal{D}_k$  and base  $S$ , we may then construct the corresponding bundle with fibre  $Z \in \mathcal{Q}_k$  and the same base.

### 2.2.2. $\cup_{j < k} \mathcal{Q}_j + \mathcal{I}_k \rightsquigarrow \mathcal{D}_k$

Fix  $(Y_0, h_0) \in \mathcal{I}_k$ . Our goal is to define the resolution blowup  $Y$  (or  $Y_\epsilon$ ) of  $Y_0$  using an appropriate collection of QAC spaces  $Z_j$ ,  $j < k$ . Not every such  $Y_0$  is smoothable, so we must choose one that is. The precise criterion for smoothability has a slight twist since we wish to ensure that the smoothing respects the local fibration structures,

and so is defined inductively. The condition is vacuous when  $k = 0$ . When  $k \geq 1$ , a neighbourhood of  $S_k$  in  $Y_0$  is a cone bundle with (singular) fibre  $\Sigma$ . Assume that we have already chosen an appropriate smoothing of  $Y_0 \setminus S_k$ , which we denote  $\tilde{Y}^{(k)}$ ; this is an open manifold with end identified with a bundle of truncated cones over  $S_k$  with link a smooth manifold  $\tilde{\Sigma}$ , which is itself a resolution blowup of  $\Sigma$ . Set  $Y' = \partial\tilde{Y}^{(k)}$  (this is of course the bundle over  $S_k$  with fibre  $\tilde{\Sigma}$ ). We now require that  $\tilde{\Sigma}$  is the boundary of a smooth compact manifold  $K$ ; the final smoothing of  $Y_0$  is obtained by replacing each truncated cone in this bundle with a copy of  $K$ , identifying  $\partial K$  with  $\tilde{\Sigma}$  on each fibre. The smoothing at each step is highly nonunique, and earlier choices may obstruct smoothability at a lower depth, but we fix the choices below.

In order to modify this description to obtain the resolution blowup of  $Y_0$  we must show how to incorporate the degeneration parameter  $\epsilon$ . Let  $S_j$  be the union of singular strata of depth  $j$  in  $Y_0$ . Assume by induction that we have chosen  $Z_{j-1} \in \mathcal{Q}_{j-1}$  to resolve the cone bundle over  $S_j$ , and hence have defined a resolution blowup  $(\tilde{Y}_\epsilon^{(k)}, h_\epsilon^{(k)})$  of  $\tilde{Y}^{(k)}$ . Note that this is done ‘in reverse order’, i.e. we first resolve the singularity at  $S_{k-1} \setminus S_k$  and then continue in order of decreasing depth. The radial variable  $r$  for  $Y_0$  at  $S_k$  can be lifted to a radial variable near the end of  $\tilde{Y}_\epsilon^{(k)}$ . We excise the region  $r \leq 1$ ; the resulting smoothed manifold with boundary has boundary  $V_\epsilon$ , which is a bundle over  $S_k$  with fibre  $Y'_\epsilon$ . Denote by  $h'_\epsilon$  the restriction of  $h_\epsilon^{(k)}$  to each of these fibres; thus  $(Y'_\epsilon, h'_\epsilon) \in \mathcal{D}_{k-1}$ . Finally, suppose that  $(Z, g_Z)$  is an element of  $\mathcal{Q}_{k-1}$  with link  $(Y'_\epsilon, h'_\epsilon)$ . We attach these spaces in fibrewise in the bundle  $V_\epsilon$  using the same radial identification map exactly as above. This completes the construction of  $Y_\epsilon$ .

As for the metric, we can certainly assume that  $h_\epsilon^{(k)}$  extends to  $\{\epsilon < r \leq 1\} \times V_\epsilon$  with the form  $dr^2 + r^2 h'_{\epsilon/r} + \kappa$ . Since the metric on  $Z$  has the form  $g_Z \sim d\rho^2 + \rho^2 h'_{1/\rho}$ , we can join the metrics  $h_\epsilon^{(k)}$  and  $\epsilon^2 g_Z$  just as for  $k = 1$  via the identification  $\epsilon\rho = r$  to get a metric  $h_\epsilon$ . The resulting space  $Y = Y_\epsilon$  is a smooth manifold, and altogether we obtain the element  $(Y_\epsilon, h_\epsilon) \in \mathcal{D}_k$ .

### 2.2.3. Resolved total boundary defining function

Choose  $Y_0 \in \mathcal{I}_k$ , with radial blowup  $\bar{Y}_0$ , and fix also a resolution blowup  $Y_\epsilon \rightarrow Y_0$ . The codimension one boundary faces of  $\bar{Y}_0$  are in bijective correspondence with the singular strata of  $Y_0$ . Denote by  $r_j$  a defining function for the (possibly disconnected) boundary hypersurface corresponding to the stratum  $S_j$ . The product  $R = r_1 \dots r_k$  is called a ‘total boundary defining function’; it is positive precisely on the open dense principal stratum  $S_0$ , and pushes forward to a continuous function on  $Y_0$ .

Our aim in this subsection is to construct a family of weight functions  $R_\epsilon$  on  $Y_\epsilon$  converging to  $R$  with  $\epsilon$ . The motivation is as follows. The natural elliptic estimates on  $Y_0$  for a generalized Laplacian are those involving Sobolev or Hölder spaces weighted by various powers of the  $r_j$ . For the particular arguments in this paper, it is sufficient to use powers of  $R$  as weight functions. The approximations  $R_\epsilon$  will be used to properly phrase estimates for generalized Laplacians on  $Y_\epsilon$  which are uniform in  $\epsilon$ . We shall not need to state the most precise sense in which  $R_\epsilon \rightarrow R$ ; at first it suffices to know that this convergence holds locally uniformly on compact subsets of the principal open stratum of  $Y_0$ , but in the course of the construction it will be made clear that these functions have precise limiting behaviour near the various

singular strata as  $\epsilon \rightarrow 0$ . To obtain the most satisfactory and precise description of these functions, we regard  $R_\epsilon$  as a single function on

$$\mathcal{Y}_0 = Y_0 \times \{0\} \cup \bigcup_{0 < \epsilon \leq 1} Y_\epsilon \times \{\epsilon\}.$$

Note that  $\mathcal{Y}_0$  is metrically complete with respect to  $G = d\epsilon^2 + h_\epsilon$ . Its singularities may be resolved by radial blowup of the singular strata at  $\epsilon = 0$ , as usual in order of decreasing depth of the singularity. The space obtained in this way is denoted  $\mathcal{Y}$  and called the single space of this resolution blowup. This is a manifold with corners; its various boundary faces at  $\epsilon = 0$  correspond to the compactifications of the various QAC components of the resolution blowup  $Y_\epsilon \rightarrow Y_0$ , or to the resolved single spaces of the  $(\mathcal{D}_j, j < k)$  links of these QAC spaces. The most precise description of  $\{R_\epsilon\}$  then is that it is a polyhomogeneous function with specific behaviour at each of these boundary faces of  $\mathcal{Y}$ . We do not pursue this description here since relatively little about it is needed below, but this point of view is developed and explained more carefully in [4].

Parenthetically, the name ‘single space’ may not seem particularly descriptive, but in further developments of this subject it is meant to indicate that  $\mathcal{Y}$  has two counterparts, the double and triple spaces, which are needed to study the uniform behaviour of degenerating differential and pseudodifferential operators on  $Y_\epsilon$ , their composition properties, etc.; see [15] for more on all of this when  $k = 1$ .

The functions  $R_\epsilon$  are constructed, as usual, by induction on the depth. When  $Y \in \mathcal{D}_0$ , we set  $R_\epsilon \equiv 1$ . Now suppose that  $R_\epsilon$  has been chosen on every  $(Y', h'_\epsilon) \in \mathcal{D}_j$ ,  $j < k$ , and consider some element  $(Y_\epsilon, h_\epsilon) \in \mathcal{D}_k$ . By definition, there is some family of QAC spaces  $Z \in \mathcal{Q}_{k-1}$  which pinch off in the limit as  $\epsilon \rightarrow 0$  and  $(Y_\epsilon, h_\epsilon) \rightarrow (Y_0, h_0) \in \mathcal{I}_k$ . For simplicity, assume that  $S_k$  is a single point. Recall the definition  $Y_\epsilon = \tilde{Y}_{0,1}^{(k)} \cup_{\psi_\epsilon} Z_{1/\epsilon}$ . As in the previous inductive arguments, assume that  $R_\epsilon$  has been chosen on the depth  $k - 1$  region where  $r \geq 1$  in  $\tilde{Y}_0^{(k)}$ , so we need only extend it over this final region  $Z_{1/\epsilon}$ , which is a truncated QAC space. Denote the boundary  $\{r = 1\}$  as  $(Y'_\epsilon, h'_\epsilon) \in \mathcal{D}_{k-1}$ ; this is also identified with  $(\partial Z_{1/\epsilon}, \epsilon^2 g_Z|_{Y'_\epsilon})$ . As  $\epsilon \rightarrow 0$ , this converges to  $Y_0 \setminus \{r < 1\}$ . Let  $R'_\epsilon$  be the restriction of  $R_\epsilon$  to this boundary.

To define its extension to  $Z_{1/\epsilon}$ , recall first that the restriction of the metric  $h_\epsilon$  to this region is  $\epsilon^2 g_Z$ , and that in terms of some radial function  $\rho$  on the original QAC space  $Z$ ,  $\epsilon^2 g_Z \sim dr^2 + r^2 h_{\epsilon/r}$  where  $r = \epsilon\rho$ , for  $\epsilon \leq r \leq 1$ .

The boundary  $Y'_\epsilon$  decomposes into three regions, the first where  $R'_\epsilon = \epsilon$ , the second where  $R'_\epsilon = 1$  and the third where  $\epsilon < R'_\epsilon < 1$ . Similarly, for each  $\epsilon \leq r \leq 1$ , the level set  $Z_{r/\epsilon}$  decomposes into three regions, where  $R'_{\epsilon/r} = \epsilon/r$ ,  $R'_{\epsilon/r} = 1$  and  $R'_{\epsilon/r}(q) \in (\epsilon/r, 1)$ , respectively. The union of the first regions, over  $\epsilon \leq r \leq 1$ , has the form of a product  $K_{Z'} \times [\epsilon, 1]$ , and the metric here is also of (approximate) product form. (Here  $Z'$  is the element of  $\mathcal{Q}_k$  which pinches off in  $Y'_\epsilon$  as  $\epsilon \rightarrow 0$ .) We define  $R_\epsilon = \epsilon$  on this entire region. Similarly, on the union of the third regions, where  $R'_{\epsilon/r} = 1$ , we set  $R_\epsilon = r$ . This is just the linear ‘radial’ extension from the portion of  $Y'_\epsilon$  where  $R'_\epsilon = 1$ . In the middle region, the definition is more complicated. The idea, however, is that we are regarding the union of this region and the first one as a ‘sector’ in  $Z' \times [\epsilon, 1]$  of the form  $\{(z', r) : R'_\epsilon(z') \leq r\}$ , with the product metric, and we would like  $R_\epsilon$  to be the restriction to this sector of the pullback of  $R'_\epsilon$  to this product. In other words, we simply extend  $R_\epsilon$  linearly along each level set where  $r$

is constant. The precise formula is, for  $\epsilon < r < 1$  and  $y' \in Y'_{r/\epsilon}$ ,

$$R_\epsilon(r, y') = \frac{r(rR'_{\epsilon/r}(y') - \epsilon)}{r - \epsilon} + \frac{r\epsilon(1 - R'_{\epsilon/r}(y'))}{r - \epsilon}.$$

As a check, note that when  $R'_{\epsilon/r} = \epsilon/r$ ,  $R_\epsilon(r, y') = \epsilon$ , while when  $R'_{\epsilon/r}(y') = 1$ ,  $R_\epsilon(r, y') = r$ .

This completes the extension of  $R_\epsilon$  to all of  $Y_\epsilon$ . It is not hard to show that  $R_\epsilon$  lifts nicely to the single space  $\mathcal{Y}$ , but we leave details to the interested reader.

### 3. Spectral convergence of generalized Laplacians

We now come to the main result on spectral convergence which asserts that if  $L_\epsilon$  is a generalized Laplacian on  $(Y_\epsilon, h_\epsilon)$  for  $\epsilon > 0$ , where  $(Y_\epsilon, h_\epsilon) \in \mathcal{D}_k$ , then under certain natural hypotheses, the spectrum of  $L_\epsilon$  converges to that of the Friedrichs extension of  $L_0$  on the limiting space  $Y_0 \in \mathcal{I}_k$ .

We give a fairly detailed sketch of the proof of this result. There are two things to prove: first, if  $\bar{\lambda} \in \mathbb{R}$  is an accumulation point as  $\epsilon \rightarrow 0$  of the spectrum of  $L_\epsilon$ , then  $\bar{\lambda}$  is in the spectrum of the Friedrichs extension of  $L_0$  on  $Y_0$ ; second, if  $\bar{\lambda} \in \text{spec}(L_{Y_0})$ , then it is an accumulation point of the spectrum of  $L_\epsilon$  as  $\epsilon \rightarrow 0$ . We shall restrict attention to the first implication since its converse is proved in a straightforward way by truncating the Friedrichs eigenfunctions on  $Y_0$ , transplanting these to  $Y_\epsilon$  and using minimax. This last argument is explained in detail when  $k = 1$  in [15], and in general in [4].

The argument now proceeds as follows. Suppose that  $\epsilon_j \searrow 0$  and there exists  $\lambda(\epsilon_j) \in \text{spec}(L_{\epsilon_j})$  with  $\lambda(\epsilon_j) \rightarrow \bar{\lambda}$ . Denote by  $\phi_j$  the corresponding eigenfunction; thus

$$(L_{\epsilon_j} - \lambda(\epsilon_j)) \phi_j = 0.$$

To show that  $\bar{\lambda} \in \text{spec}(L_0)$ , we prove that  $\phi_j$  converges to a nontrivial function  $\bar{\phi}$ , that  $(L_0 - \bar{\lambda})\bar{\phi} = 0$ , and that  $\bar{\phi}$  lies in the Friedrichs domain of  $L_0$ .

Since  $Y$  is compact, we can normalize by replacing each  $\phi_j$  by a constant multiple, and so assume that the sequence  $\{\phi_j\}$  is bounded in  $\mathcal{C}^3$ , for example, and then using Arzela-Ascoli, we can pass to a subsequence which converges locally uniformly on any compact subset of the principal top-dimensional stratum of  $Y_0$ . Using local elliptic estimates, this convergence is in  $\mathcal{C}^\infty$ . However, the limiting function could be identically zero. To preclude this, we choose a different normalization, using the functions  $R_\epsilon$  constructed in the last section, which also gives some information about the growth rate of the limiting function near the singular strata. The normalization is now obtained by multiplying each  $\phi_j$  by an appropriate constant so that

$$\sup_{q \in Y_\epsilon} R_\epsilon^\delta |\phi_\epsilon| = 1. \tag{1}$$

The constant  $\delta > 0$  will be chosen later in the argument. For simplicity, drop the subscript  $j$  and index  $q$ ,  $R$ ,  $\lambda$  and  $\phi$  by  $\epsilon \rightarrow 0$ .

Suppose that the supremum in (1) is attained at some point  $q_\epsilon$ . The simplest case occurs when  $q_\epsilon \rightarrow \bar{q}$  and this limit point lies in the interior of the regular part of  $Y_0$ . Since  $R_\epsilon \rightarrow R$ , we have  $\liminf_{\epsilon \rightarrow 0} R_\epsilon(q_\epsilon) > 0$ . Thus, passing to a limit in this

equation yields  $\phi_\epsilon \rightarrow \bar{\phi}$  where  $(L_0 - \bar{\lambda})\bar{\phi} = 0$  and

$$|\bar{\phi}(q)| \leq R^{-\delta} \quad \text{for all } q,$$

and moreover,  $\bar{\phi}(\bar{q}) = R(\bar{q})^{-\delta} \neq 0$ , so  $\bar{\phi} \not\equiv 0$ .

Under these conditions, if we choose  $0 < \delta < \ell/2 - 1$ , where  $\ell$  is the minimal codimension of any of the singular strata of  $Y_0$ , then  $\bar{\phi}$  is in the Friedrichs domain of  $L_0$ . This is a straightforward computation, at least when  $\ell > 2$ . When  $\ell = 2$ , one must modify the normalization slightly and work a bit harder to characterize the Friedrichs domain, cf. [4].

It must be shown that this is the only possibility. In other words, we must prove that any case in which  $R_\epsilon(q_\epsilon) \rightarrow 0$ , so that the limit  $\bar{q}$  lies on one of the singular strata of  $Y_0$ , leads to a contradiction. This has many possible subcases, depending on precisely how quickly and in what direction  $q_\epsilon$  converges to one of these strata. We shall explain this in detail, first when  $k = 1$ , then when  $k = 2$ , and then indicate the inductive argument for the general case.

**k = 1:** (This already appears in Rowlett's thesis [15]) We are assuming that  $R_\epsilon(q_\epsilon) \rightarrow 0$ , and there are two subcases, depending on whether  $R_\epsilon(q_\epsilon) \leq C\epsilon$  or  $R_\epsilon(q_\epsilon)/\epsilon \rightarrow \infty$ . In the first of these,  $q_\epsilon$  lies in some fixed compact subset of the expanding AC space  $Z_{1/\epsilon}$ . Rewriting the eigenvalue equation in this region in terms of the metric  $g_Z$  (instead of  $\epsilon^2 g_Z$ ), we get that

$$(\epsilon^{-2}L_Z - \lambda_\epsilon)\phi_\epsilon = \epsilon^{-2}(L_Z - \epsilon^2\lambda_\epsilon)\phi_\epsilon = 0.$$

Now, replace  $\phi_\epsilon$  by  $\psi_\epsilon = \epsilon^{-\delta}\phi_\epsilon$ , so that  $|\psi_\epsilon(q_\epsilon)|$  is bounded away from 0 as  $\epsilon \rightarrow 0$ . Pass to a limit, to obtain a nontrivial function  $\bar{\psi}$  defined on the entirety of  $Z$  which satisfies

$$L_Z\bar{\psi} = 0 \quad \text{and} \quad |\bar{\psi}| \leq \rho^{-\delta}.$$

By the hypothesis of the theorem, such a function cannot exist, which shows that this subcase cannot occur.

In the second situation, we perform a similar rescaling. However, now  $q_\epsilon$  lies on the expanding conic region of  $Z_{1/\epsilon}$  and escapes from every compact subset of  $Z$ . Using the usual coordinates  $(\rho, y)$  on this subset, replace  $\rho$  by  $\hat{\rho} = \rho/\rho(q_\epsilon)$ , and rescale the metric and  $\phi_\epsilon$  accordingly. The rescaled space now converges to the complete cone  $C(Y')$  (where  $Y'$  is the limiting cross-section of  $Z$ , which since  $k = 1$  is simply a compact smooth manifold), and the limiting function  $\bar{\psi}$  satisfies

$$L_{C(Y')}\bar{\psi} = 0, \quad \text{and} \quad |\bar{\psi}| \leq \hat{\rho}^{-\delta}.$$

Provided  $\delta$  is not an indicial root of this conic operator, this too is impossible, since although there are solutions which can decay like  $\hat{\rho}^{-\delta}$  as  $\hat{\rho} \rightarrow \infty$ , they must blow up at a faster rate as  $\hat{\rho} \rightarrow 0$ , and vice versa. Therefore, this subcase is also impossible. This proves the result when  $k = 1$ .

**k = 2:** This case is slightly more involved, but incorporates all the issues which appear in the general situation. For simplicity, assume that the depth 2 stratum of the limiting space  $Y_0 \in \mathcal{I}_0$  is a single point (rather than an edge); there is an element  $Z \in \mathcal{Q}_1$  which is pinched off in the limit, and it has link at infinity  $Y'_0 \in \mathcal{I}_1$ . The element of  $\mathcal{Q}_0$  which is pinched off along the depth 1 stratum is  $Z'$  with link the smooth manifold  $Y''_0 \in \mathcal{I}_0$ . As before, we are in the situation where  $R_\epsilon(q_\epsilon) \rightarrow 0$ . By a similar process of recentering the coordinate system, rescaling the metric and

renormalizing the function  $\phi_\epsilon$ , we can again arrange to pass to a nontrivial limit  $\bar{\psi}$ . There are now four subcases:

- i)  $\bar{\psi}$  is defined on  $Z \in \mathcal{Q}_1$  and satisfies  $|\bar{\psi}| \leq \rho^{-\delta}$ ;
- ii)  $\bar{\psi}$  is defined on  $C(Y'_0)$  and satisfies  $|\bar{\psi}| \leq \hat{\rho}^{-\delta}$ ;
- iii)  $\bar{\psi}$  is defined on the cylindrical set  $Z' \times \mathbb{R}^\ell$  and satisfies  $|\bar{\psi}| \leq \rho^{-\delta}$ , where now  $\rho$  is the radial function on  $Z'$ , so in particular this bound is uniform in the  $\mathbb{R}^\ell$  directions;
- iv)  $\bar{\psi}$  is defined on the cylindrical set  $C(Y''_0) \times \mathbb{R}^\ell$  and satisfies  $|\bar{\psi}| \leq \hat{\rho}^{-\delta}$  where  $\hat{\rho}$  is the radial function on the first conic factor.

These cases correspond, respectively, to when  $q_\epsilon$  converges to the depth 2 singular locus with distance comparable to  $\epsilon$ , to when it converges to this same locus at a slower rate while remaining away from the adjacent depth 1 edge, or to when  $q_\epsilon$  converges to this depth 1 edge more quickly than to the depth 2 locus; this last case splits into two subcases depending on whether  $R_\epsilon(q_\epsilon) \leq C\epsilon$  or not.

In order to rule out each of these cases, we require the following

**Lemma.** *Suppose that  $Z \in \mathcal{Q}_0$  and  $L_Z\phi = 0$  has no solutions which decay like  $\rho^{-\delta}$ . Lift  $\rho$  to  $Z \times \mathbb{R}^\ell$  so that it is constant with respect to the second factor. If  $L_{Z \times \mathbb{R}^\ell}\phi = 0$  and  $|\phi| \leq \rho^{-\delta}$ , then  $\phi \equiv 0$ . Similarly, if  $\Sigma$  is any smooth manifold and  $0 < \delta$  is sufficiently small, if  $L_{C(\Sigma) \times \mathbb{R}^\ell}\phi = 0$  and  $|\phi| \leq s^{-\delta}$ , where  $s$  is the radial variable on the cone, then  $\phi \equiv 0$ .*

*Proof.* Let  $W$  denote either  $Z$  or  $C(\Sigma)$ . Then

$$L_{W \times \mathbb{R}^\ell} = L_W + \Delta_{\mathbb{R}^\ell}.$$

The goal is to prove that if  $\phi$  is in the nullspace of this operator and has a uniform bound in the second  $\mathbb{R}^\ell$  factor, then it is actually independent of  $\mathbb{R}^\ell$ , so that we can regard it as a function on  $W$  and hence reduce to the hypothesis of the theorem. Let  $u$  be a linear variable in  $\mathbb{R}^\ell$ . Replacing  $\delta$  by any slightly smaller positive number, it suffices to prove that

$$L_{W \times \mathbb{R}^\ell} : \mathcal{S}'(\mathbb{R}^\ell; \rho^{-\delta+n/2}L^2(W)) \longrightarrow \mathcal{S}'(\mathbb{R}^\ell; \rho^{-\delta-2+n/2}L^2(W))$$

is injective, where  $n = \dim W$ , or equivalently, by duality, that

$$L_{W \times \mathbb{R}^\ell} : \mathcal{S}(\mathbb{R}^\ell; \rho^{\delta-n/2+2}L^2(W)) \longrightarrow \mathcal{S}(\mathbb{R}^\ell; \rho^{\delta-n/2}L^2(W))$$

is surjective. For this, take the Fourier transform in the  $\mathbb{R}^\ell$  direction. The fact that

$$L_W - |\eta|^2 : \mathcal{S}(\mathbb{R}^\ell_\eta; \rho^{\delta+n/2}L^2(W)) \longrightarrow \mathcal{S}(\mathbb{R}^\ell_\eta; \rho^{\delta-2+n/2}L^2(W))$$

is surjective for every  $\eta$  follows easily from the hypothesis that  $L_W$  is self-adjoint and nonnegative and that  $L_W$  is surjective. A right inverse for  $L_{W \times \mathbb{R}^\ell}$  is then constructed by Fourier synthesis.  $\square$

$k > 2$ : It should be clear now how to proceed to the general case. The steps are the same. The point  $q_\epsilon$  can approach the singular strata in many different ways now, but by induction, it suffices only to consider the same sort of situation as we encountered earlier in the construction of the function  $R_\epsilon$ . Namely, assume for simplicity that

the stratum of depth  $k$  in  $Y_0$  is an isolated point and suppose that  $q_\epsilon$  converges to this point. Then we may consider  $q_\epsilon$  as lying in the rescaled truncated space  $Z_{1/\epsilon}$ ,  $Z \in \mathcal{Q}_{k-1}$ . If  $q_\epsilon$  remains in a compact subset of this space, then an appropriate rescaling converges to a limiting solution of  $L_Z\psi = 0$  on all of  $Z$  which satisfies  $|\psi| \leq \rho^{-\delta}$ . However, if  $\rho(q_\epsilon) \rightarrow \infty$  then rescale by  $\hat{\rho} = \rho/\rho(q_\epsilon)$  so as to assume that the rescaled point  $\hat{q}_\epsilon$  remains on the slice  $\hat{\rho} = 1$ . This slice, however, is now  $Y'_{1/\rho(q_\epsilon)} \in \mathcal{D}_{k-1}$ , and there are various possibilities for whether  $\hat{q}_\epsilon$  approaches the singular stratum of this space, and if so, how and in what asymptotic direction. The rescalings necessary in all of these cases are covered by the inductive procedure, with the result that we always get a limiting solution  $\psi$  on a space  $W \times \mathbb{R}^\ell$  where now  $W$  is either an element of  $\mathcal{Q}_j$  or else a cone over some element  $Y''_0 \in \mathcal{I}_j$  for  $j < k$ . It is necessary to prove the extension of the Lemma to this more general setting. This is precisely the point where the main inductive hypothesis enters. Namely, we wish to simply assume that there are no decaying solutions  $\phi$  to  $L_Z\phi = 0$  on the QAC space of greatest depth which pinches off in this limit; however, in the duality argument, we then need the fact that  $L_Z$  is also surjective on spaces of sections decaying like some small power of  $\rho$  on  $Z$ . This fact follows from the Fredholm theory for  $L_Z$  since  $Z \in \mathcal{Q}_{k-1}$  has already been treated in a previous step of the big induction scheme. This finishes the proof in all cases.

## 4. Further directions

As explained in the introduction, this note is intended as an announcement of the work in [4] and a preliminary attack on the sorts of iteration arguments needed to understand the elliptic theory on these various classes of spaces. There are numerous issues which we have either discussed very briefly or omitted altogether, so these proofs are incomplete as presented here (when  $k > 1$ ). More complete explanations of these arguments and the development of the Fredholm theory in weighted Sobolev and Hölder spaces for generalized Laplacians on QAC manifolds is contained in [4].

We shall conclude with a brief discussion of the most important directions beyond the results achieved in that paper.

First, there is a very refined elliptic theory on AC spaces, i.e. for generalized Laplacians  $L_Z$  where  $Z \in \mathcal{Q}_0$ , which relies on the techniques of geometric microlocal analysis, see [11]. One obvious direction would be to replace the crude rescaling arguments used here with a parametrix construction in a pseudodifferential calculus adapted to the geometries of elements in  $\mathcal{Q}_k$  for any  $k$ . This is indeed possible, albeit a moderately daunting task (even at just a combinatorial level). Some specific problems which should be attacked are the understanding of Hodge theory, i.e. the topological identification of  $L^2$  harmonic forms, as well as Atiyah-Patodi-Singer type index theorems in this setting. Of particular interest will be the specializations of these results to the special setting of crepant resolutions of Kähler orbifolds.

Very closely related to this is the need for a refinement of the spectral convergence results for elements of  $\mathcal{D}_k$ . The proofs here give no uniformity at all. It is possible (as in [5]) to carry out similar proofs for the heat kernel restricted to times  $t \geq t_0 > 0$ , which then gives some control on the rates of convergence. However, this is not sufficient if one wishes to understand the limiting behaviour of global spectral

invariants, including determinants, eta functions, etc. This too can be approached via parametrix methods; the case  $k = 1$  appears in [15].

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