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CR MAPPINGS AND THEIR HOLOLOMORPHIC EXTENSION

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If M is a smooth manifold of real dimension $2n+1$, we say that M is a *CR manifold* of codimension one with *CR bundle* $\mathcal{V}$, if $\mathcal{V}$ is a subbundle of $\mathbb{C}TM$, the complexified tangent bundle of M, satisfying

$$dim_{\mathbb{C}} \mathcal{V} = n, \qquad \mathcal{V} \cap \overline{\mathcal{V}} = 0.$$

Any smooth real hypersurface M in $\mathbb{C}^{n+1}$ is a CR manifold of codimension one, where $\mathcal{V}$ is the subbundle of antiholomorphic tangent vectors to M.

Let $(M, \mathcal{V})$ and $(M', \mathcal{V}')$ be two CR manifolds of codimension one. A smooth mapping from M into M' is called CR if for all $p \in M$

$$H'(\mathcal{V}_p) \subset \mathcal{V}'_{H(p)}.$$

We recall the following definition introduced in Baouendi-Jacobowitz-Treves [**3**]. If M is a real analytic hypersurface in $\mathbb{C}^{n+1}$ containing the origin and defined locally by $\rho(z, \bar{z}) = 0$, $d\rho \neq 0$, we say that M is essentially finite at 0 if for any sufficiently small $z \in \mathbb{C}^{n+1} \setminus \{0\}$, there exists an arbitrarily small $\varsigma \in \mathbb{C}^{n+1}$ satisfying: $\rho(z, \varsigma) \neq 0$, $\rho(0, \varsigma) = 0$.

Our main result is the following:

THEOREM 1. *Let M and M' be real analytic hypersurfaces in $\mathbb{C}^{n+1}$ and $H : M \to M'$ a smooth CR mapping, defined near $p_0 \in M$ with $H(p_0) = p'_0$, and satisfying*

$$(1) \qquad H'(\mathbb{C}T_{p_0}M) \not\subset \mathcal{V}'_{p'_0} \oplus \overline{\mathcal{V}}'_{p'_0},$$

where $\mathcal{V}'$ is the CR bundle of M'. If M and M' are essentially finite at p_0 and p_0' respectively then H extends as a holomorphic mapping from a neighborhood of p_0 in $\mathbb{C}^{n+1}$ to $\mathbb{C}^{n+1}$.

Theorem 1 was first proved for $n = 1$ by S. Bell and the authors (see [**1**], [**2**]). It generalizes the result in the diffeomorphic case proved in [**3**]. We refer to the references of [**2**] and [**3**] for earlier works on holomorphic extendibility of CR mappings under stronger conditions.

The following is a key ingredient in the proof of Theorem 1. If j is a smooth CR function defined on M then there exists a unique formal (holomorphic) power series $J(z) = \sum a_\alpha z^\alpha$, $a_\alpha \in \mathbb{C}$, such that, if $U \ni u \mapsto Z(u) \in \mathbb{C}^{n+1}$ ($U \subset \mathbb{R}^{2n+1}, Z(0) = 0$) is a parametrization of M, then the Taylor series of $j(Z(u))$ at 0 is given by $J(Z(u))$. On the other hand it is clear that a CR mapping between two hypersurfaces M and M' in $\mathbb{C}^{n+1}$ is given by $(n+1)$ CR functions $(j_1, \ldots, j_{n+1})$. Such a mapping is called of *finite multiplicity* at 0 if

$$dim_{\mathbb{C}} \mathcal{O}[[Z]]/(J(Z)) < \infty,$$

where $\mathcal{O}[[Z]]$ is the ring of formal power series in $(n+1)$ indeterminates and $(J(Z))$ is the ideal generated by $(J_1(Z), \ldots, J_{n+1}(Z))$. Here the dimension is taken in the sense of vector spaces. We have the following:

THEOREM 2. *If M and M' are essentially finite at p_0 and p_0' respectively then a CR mapping $H : M \to M'$ is of finite multiplicity at p_0 if and only if condition (1) of Theorem 1 holds.*

We may restate Condition (1) in terms of local coordinates. We may assume $p_0' = H(p_0) = 0$ and M and M' are given locally by

$$Im\ w = \varphi(z, \bar{z}, \text{Re } w), \qquad Im\ w = \psi(z, \bar{z}, \text{Re } w) \tag{2}$$

with $\varphi(z, 0, \text{Re } w) = \psi(z, 0, \text{Re } w) = 0$; $z \in \mathbb{C}^n$, $w \in \mathbb{C}$. The map H is then given by $n+1$ CR functions $(f_1, \ldots, f_n, g) = (f, g)$ defined on M. Therefore we have

$$\frac{g - \bar{g}}{2i} = \psi(f, \bar{f}, \frac{g + \bar{g}}{2}). \tag{3}$$

With this notation Condition (1) is equivalent to

$$\frac{\partial g}{\partial s}(0) \neq 0, \tag{4}$$

with $s = \operatorname{Re} w$. (Here f_j and g are considered as smooth functions of z, $\overline{z}$, s).

Using Theorem 1 as well as Diederich-Fornaess [**5**], [**6**], Fornaess [**7**] and Bell-Catlin [**4**], we obtain the following

THEOREM 3. *Let D and D' be two bounded pseudoconvex domains in $\mathbb{C}^{n+1}$ with real analytic boundaries and $H : D \to D'$ a proper, holomorphic mapping. Then H extends holomorphically to a neighborhood of $\overline{D}$, the closure of D.*

We give here an outline of the proof of Theorem 1. By solving (3) for $\overline{g}$ we obtain a holomorphic function Q

$$\overline{g} = Q(f, \overline{f}, g). \tag{5}$$

As in [**3**] by writing

$$Q(f, \lambda, g) = \sum Q_{\varsigma^\alpha}(f, \overline{f}, g) \frac{(\lambda - \overline{f})^\alpha}{\alpha!}$$

we are reduced to showing that for $z_0 \in \mathbb{C}^n$ fixed, $|z_0| < r$,

$$Q_{\varsigma^\alpha}(f(z_0, \overline{z}_0, s), \overline{f}(z_0, \overline{z}_0, s), g(z_0, \overline{z}_0, s))$$

extends as a holomorphic function in $s + it$, $|s| < r$, $-R < t < 0$, for some r, R positive, and satisfies

$$|Q_{\varsigma^\alpha}| \leq C^{\alpha+1}\alpha!, \qquad C > 0. \tag{6}$$

The main ingredients used in proving the above are the following.

LEMMA 1. *If j is a smooth CR function defined on M then the Taylor series of j in the coordinates (z,s) is given uniquely by*

$$j \sim \sum a_{\alpha k} z^\alpha w^k|_{w=s+i\varphi(z,\bar z,s)}, \qquad a_{\alpha k} \in \mathbb{C}. \tag{7}$$

A basis for the CR vector fields on M is given by

$$L_j = \frac{\partial}{\partial \bar z_j} - i\frac{\varphi_{\bar z_j}}{1+i\varphi_s}\frac{\partial}{\partial s}, \qquad 1 \le j \le n, \tag{8}$$

LEMMA 2. *If $j(z,\bar z,s)$ is a CR function on M, then for all multi-indices α*

$$\overline{L}^\alpha j(0) = \left(\frac{\partial}{\partial z}\right)^\alpha J(0,0),$$

where $J(z,w) \sim \sum a_{\alpha k} z^\alpha w^k$ is as defined in Lemma 1.

Using the Nullstellensatz we may prove the following.

LEMMA 3. *For $j = 1,\dots,n$ let $F_j(z,w)$ be the formal power series associated to f_j as in Lemma 1. Let $\mathcal{I}$ be the ideal generated by $F_j(z,0)$, $1 \le j \le n$, the ring $\mathcal{O}[[Z]]$ of formal power series in the indeterminates $z_1,\dots,z_n$. Then under the assumptions of Theorem 1,*

$$\dim_{\mathbb{C}} \mathcal{O}[[z]]/\mathcal{I} < \infty, \tag{9}$$

and therefore

$$\det\left(\frac{\partial F_k}{\partial z_j}(z,0)\right) \not\equiv 0. \tag{10}$$

An immediate consequence of Lemmas 2 and 3 is that there exists a multi-index α such that

$$\overline{L}^\alpha(\det(\overline{L}_j f_k))(0) \neq 0. \tag{11}$$

LEMMA 4. *For every multi-index α and every z_0, $|z_0| < r$ there exist functions $a(s)$, $b(s)$ holomorphic in the domain $\mathcal{R} = \{s+it; |s| < r, -R < t < 0\}$, smooth in $\overline{\mathcal{R}}$ such that*

$$Q_{\zeta^\alpha}(f,\overline{f},g)(z_0,s) = \frac{a(s)}{b(s)}.$$

Lemma 4 is proved by applying successively $\overline{L}^\beta$ to (5) and using (11).

LEMMA 5. *For each j, $1 \leq j \leq n$, f_j satisfies a polynomial equation of the form*

$$f_j^{N_j} + a^j_{N_j-1} f_j^{N_j-1} + \cdots + a^j_0 = 0,$$

where $a^j_k = a^j_k(L^\gamma \overline{f}, L^\gamma \overline{g})$ is a holomorphic function of the $L^\gamma \overline{f}$, $L^\gamma \overline{g}$, for $|\gamma| \leq \gamma_0$.

The proof of Lemma 5 uses Lemma 3, as well as repeated applications of the Weierstrass Preparation theorem and the Nullstellensatz.

LEMMA 6. *There exists N such that for each multi-index α, $Q_{\varsigma^\alpha}(f, \overline{f}, g)(z, \overline{z}, s)$ is a root of a polynomial of the form*

$$X^N + b^\alpha_{N-1} X^{N-1} + \cdots + b^\alpha_0 = 0 \tag{12}$$

where the b^α_k are holomorphic functions of $L^\gamma \overline{f}$ and $L^\gamma \overline{g}$, $|\gamma| \leq \gamma_0$, and satisfies

$$|b^\alpha_j(L^\gamma \overline{f}, L^\gamma \overline{g})| \leq (C^{\alpha+1} |\alpha|!)^{N-j} \tag{13}$$

at $(z, \overline{z}, s + it)$ for $|z| < r$, $|s| < r$ and $-R \leq t \leq 0$.

From Lemmas 4 and 6 it follows, using the Lemma in [**2**], that each $Q_{\varsigma^\alpha}(f, \overline{f}, g)$ extends holomorphically to $\mathcal{R}$. Finally, the estimate (6) follows from (13).

For higher codimension, a slight modification of the proof of Theorem 1 yields the following.

THEOREM 4. *Let M and M' be real analytic generic CR submanifolds of real codimensional ℓ in $\mathbb{C}^{n+\ell}$ and $H : M \to M'$ a smooth CR mapping defined near $p_0 \in M$, $H(p_0) = p_0'$, and satisfying*

$$\dim_{\mathbb{C}}(H'(\mathbb{C}T_{p_0}M)/\mathcal{V}'_{p_0} \oplus \overline{\mathcal{V}}'_{p_0'}) = \ell \tag{14}$$

where $\mathcal{V}'$ is the CR bundle of M'. Assume that M and M' are essentially finite at p_0, and that near p_0, H extends holomorphically to a wedge of edge M. Then H extends as a holomorphic mapping from a neighborhood of p_0 in $\mathbb{C}^{n+\ell}$ to $\mathbb{C}^{n+\ell}$.

Complete details of the proofs will appear elsewhere.

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