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PROPAGATION OF SINGULARITIES AND LOCAL SOLVABILITY IN GEVREY CLASSES

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The propagation of the Gevrey singularities has been investigated recently by many authors (see for example Cattabriga - Zanghirati [2] and the references there). Here we shall report on some results obtained in collaboration with Zanghirati [7] and Liess [5] concerning propagation of Gevrey singularities for pseudo differential operators with multiple characteristics; we shall also consider the strictly related problem of the Gevrey local solvability, already discussed in Rodino [6].

Let us denote by $G^s(\Omega)$ the Gevrey class of order s, $1 < s < \infty$, in the open subset Ω of $\mathbb{R}^n$. Let us write $G_0^s(\Omega) = G^s(\Omega) \cap C_0^\infty(\Omega)$; the space of the s-ultradistributions $G_0^{(s)'}(\Omega)$ and the space of the s-ultradistributions with compact support $G^{(s)'}(\Omega)$ are then defined as the duals of $G_0^s(\Omega)$, $G^s(\Omega)$, respectively. We shall also use the standard notion of Gevrey wave front set of order s of $v \in G^{(s)'}(\Omega)$, $WF_s\, v \subset \Omega \times (\mathbb{R}^n \setminus 0)$.

Our arguments will be microlocal in a small conic neighborhood Γ of a point $(x_0, \xi_0) \in \Omega \times (\mathbb{R}^n \setminus 0)$; we shall consequently refer to the factor-space of the s-microfunctions in Γ

$$M^s(\Gamma) = G_0^{(s)'}(\Omega)/\sim ,$$

where $f \sim g$ means that $\Gamma \cap WF_s(f-g) = \emptyset$.

Let us consider a classical analytic pseudo differential operator $P = p(x,D)$ with symbol

$$p(x,\xi) \sim \sum_{j=0}^{\infty} p_{m-j}(x,\xi)$$

defined in a conic neighborhood of (x_0, ξ_0). We shall assume the principal part $p_m(x,\xi)$ satisfies for some integer $k \geq 2$ the following condition

(1) <u>we may write</u> $p_m(x,\xi) = q_{m-k}(x,\xi)\, a_1(x,\xi)^k$, <u>where</u> $q_{m-k}(x,\xi)$ <u>is an elliptic symbol homogeneous of order</u> $m-k$, <u>and the first order symbol</u> $a_1(x,\xi)$ <u>is real valued and of principal type, i.e.</u> $d_{x,\xi}\, a_1(x,\xi)$ <u>never vanishes and it is not parallel to</u> $\sum_h \xi_h\, dx_h$ <u>on</u> $\Sigma = \{(x,\xi) \in \Gamma,\ a_1(x,\xi) = 0\} \neq \emptyset$.

This is equivalent to say that our operator P can be reduced, by conjugation with analytic Fourier integral operators and multiplication by elliptic factors, to the form

$$P = D^k_{x_n} + \underline{\text{pseudo differential operators of order}}$$
$$\leq k-1 .$$

The hypothesis (1) is sufficient to conclude non-analytic hypoellipticity of P and propagation of the analytic wave front set along the bicharacteristic strips associated to P (see Bony-Shapira [1]), whereas to obtain a similar result in the C^∞ category it is necessary to add the so-called Levi condition on the lower order terms (see Chazarain [3]). A natural interpolation of these results can be expressed in the frame of the Gevrey classes under the following ρ-Levi condition, $0 < \rho < 1$:

(2) Let A be a classical analytic ps.diff.operator whose principal symbol is given by the function $a_1(x,\xi)$ in (1); then P can be written in the form $P = \Sigma^k_{j=0} Q_j A^{k-j}$, where Q_j, $j = 0,\dots,k$, are classical analytic pseudo differential operators of order $\leq m-k+\rho j$.

If in (2) we set $\rho = 0$, we obtain the standard C^∞ Levi condition; in the other limit case $\rho = 1$, nothing is imposed on the lower order terms.

An operator P satisfying (1) and (2) is microlocal ly equivalent to the model

(3) $$P = D^k_{x_n} + \sum_{j=1}^{k} Q_j \, D^{k-j}_{x_n} \quad ,$$

where the Q_j , $j = 1,\dots,k$, are here classical analytic pseudo differential operators of order $\leqslant \rho j$.

THEOREM 1. (Rodino-Zanghirati [7]). Let (1), (2) be satisfied and let s be any real number with $1 < s < 1/\rho$. Write γ_0 for the bicharacteristic strip thr. ough $(x_0,\xi_0) \in \Sigma$ (we may define γ_0 to be integral curve of the Hamiltonian vector field H_{a_1} , with $a_1(x,\xi)$ as in (1), (2)).
Then, taking a sufficiently small neighborhood Γ of (x_0,ξ_0):

(i) There exists $\upsilon \in M^s(\Gamma)$ with $P\upsilon = 0$ and $WF_s\,\upsilon = \gamma_0$.

(ii) If υ is in $M^s(\Gamma)$ with $P\upsilon = 0$, then $(x_0,\xi_0) \in WF_s\,\upsilon$ implies $\gamma_0 \subset WF_s\,\upsilon$.

(iii) For every $v \in M^s(\Gamma)$ there exists $\upsilon \in M^s(\Gamma)$ such that $P\upsilon = v$.

For the proof we may refer to the model (3); its study can be further reduced to that of the first order operator:

(4) $$D_{x_n} + \lambda(x,D_x) \quad ,$$

where $\lambda(x,D_x)$ is a $k \times k$-matrix of pseudo differential operators of order $\leq \rho$. We then construct two matrices B^+, B^- of linear maps from $M^s(\Gamma)$ to $M^s(\Gamma)$, one inverse of the other, which are s-microlocal and satisfy

$$(5) \qquad B^-(D_{x_n} + \lambda(x,D_x))\, B^+ = D_{x_n} \, .$$

In this way we are reduced to prove the theorem for $P = D_{x_n}$, and that is trivial. The formal construction of $B^\pm$ as pseudo differential operators is easy by solving transport equations. However, the symbols which one obtains have an exponential growth and to give a precise meaning to $B^\pm$ we have to refer to a suitable theory of Gevrey infinite order operators (cf. Cattabriga-Zanghirati [2]).

Under the assumptions (1), (2), the conclusions of Theorem 1 fail in general for $1/\rho \leq s < \infty$ and the study of the corresponding G^s regularity requires then a further analysis of the operators Q_j in (2), (3).

We shall illustrate the new phenomena which may occur by arguing on the model (4). For sake of simplicity, we shall suppose here $\lambda(x,D_x)$ is a scalar operator with symbol

$$\lambda(x,\xi) = \lambda_\rho(x,\xi') + \lambda_0(x,\xi) \, ,$$

where $\lambda_{\rho}(x,\xi')$ is homogeneous of order ρ , $0 < \rho < 1$, with respect to $\xi' = (\xi_1,\dots,\xi_{n-1})$ and $\lambda_0(x,\xi)$ is a classical analytic symbol of order zero. Our arguments will be microlocal in a neighborhood of a point (x_0,ξ_0) with $\xi_0 = (\xi'_0,0)$.
For the operator

$$P = D_{x_n} + \lambda_{\rho}(x,D_{x'}) + \lambda_0(x,D) \tag{6}$$

the conclusions of Theorem 1 (non-hypoellipticity, propagation, local solvability) apply when $1 < s < 1/\rho$, whereas for $1/\rho \leqslant s < \infty$ we have:

<u>Theorem</u> 2 . <u>Assume</u> $\operatorname{Im} \lambda_{\rho}(x_0,\xi'_0) \neq 0$. <u>Then for</u> $1/\rho \leqslant s < \infty$ <u>the operator</u> P <u>in</u> (6) <u>is</u> G^s-<u>hypoelliptic in a neighborhood</u> Γ <u>of</u> (x_0,ξ_0) , <u>i.e.</u>

$$WF_s \; Pv = WF_s \; v \quad \underline{\text{for all}} \quad v \in M^s(\Gamma) \; ,$$

<u>and the solvability property</u> (iii) <u>in Theorem</u> 1 <u>is still valid</u>.

In fact, a parametrix P' of P can be easily constructed, $P'P = PP' =$ <u>identity</u> on $M^s(\Gamma)$, $1/\rho \leqslant s < \infty$, with symbol in a Gevrey version of the class $S^{\rho}_{\rho,0}$ of Hörmander (see for example Liess-Rodino [4]).

<u>Theorem</u> 3. <u>Assume</u> $\lambda_\rho(x,\xi')$ <u>is real valued in a conic neighborhood of</u> (x_0,ξ'_0). <u>All the conclusions of Theorem</u> 1 <u>are valid for</u> P <u>in</u> (6) <u>also when</u> $1/\rho \leqslant s < \infty$.

This a consequence of a much more general result in Liess-Rodino [5], concerning Gevrey propagation for operators of non-homogeneous type. Precisely, under the assumption in Theorem 3, we may construct Fourier integral operators $B^{\pm}$, with non-homogeneous (real) phase function, for which (5) is satisfied on $M^s(\Gamma)$, $1/\rho \leqslant s < \infty$; in this way we are again reduced to the trivial study of the operator $P = D_{x_n}$.

When $\lambda_\rho(x,\xi')$ takes values in the complex domain, but $\text{Im}\,\lambda_\rho(x,\xi')$ vanishes at (x_0,ξ'_0), then the solvability property (iii) in Theorem 1 may fail for $1/\rho < s < \infty$.
A representative example in this connection is given by the model in $\mathbb{R}^2$

$$(7) \qquad P_\rho = D_{x_2} + i x_2^h \, |D_{x_1}|^\rho \ ,$$

where h is an odd integer and $0 < \rho < 1$; the symbol $\lambda_\rho = i x_2^h |\xi_1|^\rho$ is here considered in a neighborhood of $x_0 = (0,0)$, $\xi_0 = (1,0)$.

<u>Theorem</u> 4. <u>Assume</u> $1/\rho < s < \infty$. <u>Then there exists</u>

$v \in M^s(\mathbb{R}^2)$ <u>such that</u> $(x_0, \xi_0) \in WF_s(v - P_\rho \upsilon)$ <u>for all</u> $\upsilon \in M^s(\mathbb{R}^2)$.

The theorem is proved in Rodino [6] by considering the Fourier integral operator

$$\Pi_\rho f(x) = \iint_{\vartheta > 0} e^{i\omega(x,y,\vartheta)} \vartheta^{\rho/(h+1)} f(y)\, dy\, d\vartheta$$

with non-homogeneous complex phase function

$$\omega(x,y,\vartheta) = \vartheta(x_1 - y_1) + i\vartheta^\rho (x_2^{h+1} + y_2^{h+1})/(h+1) \ .$$

The operator Π_ρ maps $G_0^s(\mathbb{R}^2)$ into $G^s(\mathbb{R}^2)$, and $G^{(s)'}(\mathbb{R}^2)$ into $G_0^{(s)'}(\mathbb{R}^2)$, for $1/\rho < s < \infty$. For the same values of s, the operator Π_ρ is s-microlocal, so it is well defined on the s-microfunctions in a neighborhood of the origin, and we also have:

$$\Pi_\rho P_\rho = 0 \ .$$

Il we take $v \in M^s(\mathbb{R}^2)$ such that $(x_0, \xi_0) \in \Pi_\rho v$, then we obtain $(x_0, \xi_0) \in WF_s(v - P_\rho \upsilon)$ for all $\upsilon \in M^s(\mathbb{R}^2)$; in fact $P_\rho \upsilon = v$ in a conic neighborhood of (x_0, ξ_0) would imply

$$\Pi_\rho (P_\rho \upsilon - v) = \Pi_\rho P_\rho \upsilon - \Pi_\rho v = \Pi_\rho v = 0$$

in the same neighborhood.

If we limit ourselves to the local point of view, the proceeding shows that for $1/\rho < s < \infty$ the operator P_ρ is <u>non-s-locally solvable</u> at $x_0=(0,0)$, i.e. there exists $f \in G_0^s(\mathbb{R}^2)$ such that the equation $P_\rho\, v = f$ has no solution $v \in G_0^{(s)'}(\mathbb{R}^2)$ in any neighborhood of the origin. In view of the obvious inclusions $G_0^s(\mathbb{R}^2) \subset C_0^\infty(\mathbb{R}^2)$, $D'(\mathbb{R}^2) \subset G_0^{(s)'}(\mathbb{R}^2)$, we have in particular that P_ρ is <u>non-locally solvable</u> in the standard C^∞ sense.
However, a solution v of the equation $P_\rho\, v = f \in C^\infty(\mathbb{R}^2)$ always exists if we allow v to be in $G_0^{(s)'}(\mathbb{R}^2)$ with $1 < s < 1/\rho$ (This follows from the local version of (iii) in Theorem 1). It is worth particularizing the computations of Rodino-Zanghirati [7] for the operator P_ρ in (7), to see explicitly how "unsolvable equations can be solved" in an ultra-distribution sense. We have to consider the pseudo differential operators

$$B_\rho^\pm f(x) = (2\pi)^{-2} \int e^{ix\xi}\, b_\rho^\pm(x,\xi)\hat{f}(\xi)d\xi$$

with infinite order symbols

$$b_\rho^\pm(x,\xi) = \exp\ [\pm\, x_2^{h+1}\, |\xi_1|^\rho/(h+1)]\ .$$

They are one inverse of the other and satisfy the identity (5), i.e.:

$$B^{-}_{\rho} \; P_{\rho} \; B^{+}_{\rho} = D_{x_2} \; .$$

Therefore a solution of $P_{\rho} \, v = f \in C_0^{\infty} (\mathbb{R}^2)$ is obtained by considering

$$\tilde{f}(x) = i \int_0^{x_2} B^{-}_{\rho} \, f(x_1, y_2) \, dy_2 \; ,$$

which is still a C^{∞} function, and setting finally $v = B^{+}_{\rho} \tilde{f}$ (which is in general a true ultradistribution in $G_0^{(s)'}(\mathbb{R}^2)$, $1 < s < 1/\rho$) . For a more detailed discussion of the problem of the Gevrey-local solvability, we refer to Rodino [6].

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