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ON THE LAX & PHILLIPS SCATTERING
THEORY FOR TRANSPORT EQUATION

H. EMAMIRAD

Let Ω be a bounded star shaped domain in \mathbb{R}^n and ρ the radius of a ball B_ρ around the origin which contains all the points of Ω . E_+ the free forward and E_- the free backward point sets are defined as follow:

$$E_+ = \{ (x,v) \in \mathbb{R}^n \times V \mid x \cdot v \geq \rho \}$$

$$E_- = \{ (x,v) \in \mathbb{R}^n \times V \mid x \cdot v \leq -\rho \}$$

Corresponding to the subsets E_\pm one defines the incoming subspace D_- and outgoing subspace D_+ by:

$$D_\pm = \{ f \in X \mid \text{supp } f \subset E_\pm \}$$

In [1] Lax & Phillips have taken the unit sphere in \mathbb{R}^n as the velocity space V and $X = L^2(\mathbb{R}^n \times V)$. They have shown that for $U_0(t)$ the one-parameter unitary group defined by $U_0(t)f(x,v) = f(x-vt,v)$ one has

THEOREM 1. The subspaces D_+ and D_- satisfy the following properties:

i)₊ $U_0(t)D_+ \subset D_+$ for $t \geq 0$

i)₋ $U_0(t)D_- \subset D_-$ for $t \leq 0$

ii) $\bigcap_{t \in \mathbb{R}^n} U_0(t)D_\pm = \{0\}$

iii) $\bigcup_{t \in \mathbb{R}^n} U_0(t)D_\pm$ is dense in X .

This theorem can be easily generalized to the case when V is an annulus contained in the unit ball of \mathbb{R}^n

$$V = \{ v \in \mathbb{R}^n \mid 0 < v_m \leq |v| \leq 1 \}$$

and $X = L^p(\mathbb{R}^n \times V)$ for $1 \leq p < \infty$.

$U_0(t)$ is a strongly continuous positive group generated by free collision transport operator

$$T_0 f = -v \cdot \nabla_x f$$

in any $L^p(\mathbb{R}^n \times V)$. For any λ in \mathcal{C} the only function which verifies $T_0 \phi = \lambda \phi$ is

$$\phi(x, v) = g(x_{\perp}, v) \exp\{-\lambda x \cdot v / |v|^2\}$$

where $x_{\perp} = x - |v|^{-2}(x \cdot v)v$. Hence for any g in $L^\infty(\mathbb{R}^n \times V)$, belongs also to $L^\infty(\mathbb{R}^n \times V)$ if and only if $\lambda = i\beta$ for any real β .

This shows that the nature of the spectrum of T_0 depends on the exponent p in L^p . In fact if we denote by $\Sigma(T_0)$ the spectrum of T_0 , using $\Sigma_p(T_0)$, $\Sigma_c(T_0)$ and $\Sigma_r(T_0)$ to denote respectively the point spectrum, continuous spectrum and residual spectrum of T_0 , we can prove the following peculiar result:

THEOREM 2. a) $\Sigma(T_0) = \Sigma_r(T_0) = i\mathbb{R}$ in $L^1(\mathbb{R}^n \times V)$.

b) $\Sigma(T_0) = \Sigma_c(T_0) = i\mathbb{R}$ in $L^2(\mathbb{R}^n \times V)$.

c) $\Sigma(T_0) = \Sigma_p(T_0) = i\mathbb{R}$ in $L^\infty(\mathbb{R}^n \times V)$.

One of our major aim in this paper is to show when the Lax & Phillips representation theorem (Theorem 1.) is valid in $L^1(\mathbb{R}^n \times V)$ for collision dynamics $U(t)$ the one parameter group generated by linearized Boltzmann operator

$$T f = -v \cdot \nabla_x f - \sigma_a(x, v) f + \int_V k(x, v', v) f(x, v') dv'$$

where σ_a and k are two non-negative measurable functions on $\mathbb{R}^n \times V$ and $\mathbb{R}^n \times V \times V$ respectively. We define the production cross section σ_p by:

$$\sigma_p(x, v) = \int_V k(x, v, v') dv'$$

and we suppose that the transport system

$$\frac{\partial u}{\partial t} = T u, \quad u(x, v, 0) = u_0(x, v) \in L^1(\mathbb{R}^n \times V)$$

is admissible . i.e:

- i) σ_a and σ_p belong to $L^{\infty}_+(\mathbb{R}^n \times V)$
- ii) There is a compact set K in Ω so that σ_a and σ_p vanish if $x \notin K$.

In the Lax & Phillips representation theorem the crucial point is the density property iii) . This property is closely related to the local decay property of the dynamics (see [2]). i.e For any compact subset K of \mathbb{R}^n and any function f in $L^1(\mathbb{R}^n \times V)$

$$(LD) \quad \int_{K \times V} |U(t)f(x, v)| dx dv \rightarrow 0$$

as $|t| \rightarrow \infty$. It comes out that this last property is also intimately related with the spectral configuration of the infinitesimal generator T of $U(t)$. In fact one can never get (LD) if $\sigma_p(T) \neq \emptyset$. The following theorem shows that this may happen to our case.

THEOREM 3. In $L^1(\mathbb{R}^n \times V)$, $\Sigma(T) = i \mathbb{R} \cup \Sigma_p(T)$ where $\Sigma_p(T)$ is either empty or at most a finite set of isolated points lying in the strip $\Lambda = \{ z \in \mathbb{C} \mid -c \leq \text{Re } z \leq c_2 \}$.

Sketch of the proof. Let us denote the operators A_1 and A_2 on $L^1(\mathbb{R}^n \times V)$ by:

$$[A_1 f](x, v) = -\sigma_a(x, v) f(x, v)$$

$$[A_2 f](x, v) = \int_V k(x, v', v) f(x, v') dv'$$

Put $A = A_1 + A_2$ and $T_1 = T_0 + A_1$. Since A_1 and A_2 are bounded by $c_1 = \|\sigma_a\|_\infty$ and $c_2 = \|\sigma_p\|_\infty$ respectively. From the theory of semigroups one can deduce that $T = T_0 + A$ generates an one-parameter group $U(t)$ and $\Sigma(T)$ lies in Λ with $c = c_1 + c_2$. Let $L_\lambda = (\lambda - T_1)^{-1} A_2$. By virtue of Dunford-Pettis theorem one can show that $\lambda \rightarrow L_\lambda^2$ is an analytic compact operator-valued function in \mathbb{C} , and we have for $\operatorname{Re} \lambda \neq 0$, $\|L_\lambda^2\| \leq \|A_2\|^2 / |\operatorname{Re} \lambda|^2$. Hence L_λ^2 tends to zero as $|\operatorname{Re} \lambda| \rightarrow \infty$. Therefore 1 and -1 are not the eigenvalues for all operators L_λ^2 . Thus by applying the analytic Fredholm Theorem $(I - L_\lambda^2)^{-1}$ exists, except at most a countable set of isolated points λ_k , where the function $\lambda \rightarrow (I - L_\lambda^2)^{-1}$ has a pole. From the two following algebraic identities:

$$\begin{aligned} (I - L_\lambda)^{-1} &= (I + L_\lambda)(I - L_\lambda^2)^{-1} \\ (\lambda - T)^{-1} &= (I - L_\lambda)^{-1}(\lambda - T_1)^{-1} \end{aligned}$$

it follows that for $\operatorname{Re} \lambda \neq 0$ any pole of $(I - L_\lambda^2)^{-1}$ is an eigenvalue of T . The finiteness of the number of these eigenvalues will be proved later.

Going back to the theorem 1. The proof of the assertions i) and ii) is a simple consequence of the following lemma.

LEMMA 4. For any $t \geq 0$ [$t \leq 0$] and any $f \in D_+$ [$f \in D_-$] one has

$$U(t)f = U_0(t)f .$$

Theorem 3 shows that the assertion iii) of Theorem 1 for $U(t)$ fails to be true in general case , but however we have:

THEOREM 5. The following assertions are equivalent

- a) $\bigcup_{t \in \mathbb{R}} U(t)D_{\pm}$ is dense in $L^1(\mathbb{R}^n \times V)$
- b) The local decay property (LD) holds .
- c) The operator T admits neither eigenvalues on the complex plane nor resonances on the imaginary axis.

For implication b) \Rightarrow a) see [3] . In order to prove c) \Rightarrow b) we have to introduce the Lax & Phillips semigroup $Z(t)$ for transport equation.

Let us define the projections P_{\pm} on $L^1(\mathbb{R}^n \times V)$ by $P_{\pm}f = \chi_{\pm}f$ where χ_{\pm} are the characteristic functions of E_{\pm} and $\chi_{\pm}' = 1 - \chi_{\pm}$. We define the Lax & Phillips semigroup by:

$$Z(t) = P_+U(t)P_-$$

Let us consider K a subspace of $L^1(\mathbb{R}^n \times V)$ consisting of functions f which are identically zero on $E_+ \cup E_-$.

THEOREM 6. The operators $\{ Z(t) \mid t \geq 0 \}$ map K into itself and form strongly continuous semigroup on K . Furthermore it is a differentiable and compact semigroup for sufficiently large t .

The eigenvalues of B the infinitesimal generator of $Z(t)$ are called *resonances* and the compactness of $Z(t)$ implies that the spectrum of B is constituted of pure resonances . Furthermore the differentiability of $Z(t)$

implies that these resonances are lying in a logarithmic region of the form

$$\Lambda = \{ \lambda \in \mathbb{C} \mid \operatorname{Re} \lambda < a - b \log |\lambda| \}$$

where a is real and $b > 0$. Thus the following theorem proves the finiteness of $\Sigma_p(T)$. This theorem is based on the fact that any eigenfunction of T vanishes out P of Ω .

THEOREM 7. $\Sigma_p(T) \subset \Sigma(B)$.

Here the fundamental problem of the existence of such resonances arises. In order to prove that $\Sigma(B) \neq \emptyset$ we will look to the interior transport problem which was posed by Jörgens [4]. He proved that in some circumstances the interior transport operator T^J admits eigenfunctions verifying the interior boundary condition:

$$\phi(x, v) = 0 \quad \text{for } x \in \partial\Omega \quad \text{and} \quad n(x) \cdot v < 0$$

where $n(x)$ is the exterior normal to Ω at x . By an extension of these eigenfunctions to whole space we prove

THEOREM 8. $\Sigma(T^J) = \Sigma(B)$.

This extension shows that the asymptotic form of these eigenfunctions look like $\exp\{-\mu x \cdot v / |v|^2\}$ when $n(x) \cdot v \geq 0$. According to Lax & Phillips terminology we will call them *generalized eigenfunctions*.

By an analysis based on a complex residues computation we prove an eigenfunction expansion for $Z(t)$ which is asymptotically valid for large t . i.e: By arranging the eigenvalues μ_j of B in decreasing order of their real parts and denote by P_j the projection into the j^{th} eigenspace and D_j^k the corresponding nilpotent operator of order k , one has

$$Z(t) \approx \sum_{\mu_j \in \Sigma(B)} e^{\mu_j t} \left(P_j + \sum_k \frac{t^k}{k!} D_j^k \right)$$

The following version of the above formula was suggested by Melrose [5],
for wave equation which is more coherent to our setting.

THEOREM 9. For any f in $L^1(\mathbb{R}^n \times V)$ there exist a sequence μ_j in \mathbb{C} and
generalized eigenfunctions $w_{j,k}$, $k = 0, \dots, m_j - 1$ such that for any $n \in \mathbb{N}$
and ε , $0 < \varepsilon < \operatorname{Re} \mu_n - \operatorname{Re} \mu_{n+1}$

$$\sup_{(x,v) \in \Omega \times V} \left| [U(t)f](x,v) - \sum_{j=1}^n e^{\mu_j t} \sum_{k=0}^{m_j-1} t^k w_{j,k}(x,v) \right| \leq c |e^{(\mu_n - \varepsilon)t}|$$

for sufficiently large t . The constant c depends only on n and ε .

This theorem yields the implication $c) \Rightarrow b)$ in theorem 5. We deduce
also from compactness of $Z(t)$ and the fact that $\{0\} \notin \Sigma_p(T)$ (see [6]) that
 $a) \Rightarrow c)$.

Finally we give a physically relevant situation in which the property $b)$
of Theorem 5 occurs. This situation is presented by Hejtmanek [7]. He
showed when the Dyson-Phillips expansion of $U(t)$ is finite, which physically
means that the system is of finite collisions then the spectrum of T does
not exceed the imaginary axis. We can conclude under the above condition
Lax & Phillips representation theorem is fully valid.

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