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Analyticity for certain solutions of nonhypoelliptic differential operators on the Heisenberg group

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We consider left invariant differential operators on the Heisenberg group G with Lie algebra $\mathfrak{g} = \mathfrak{g}_1 + \mathfrak{g}_2$ $[\mathfrak{g}_1, \mathfrak{g}_1] = \mathfrak{g}_2$, $[\mathfrak{g}_2, \mathfrak{g}] = 0$, where $X_1, X_2, \ldots, X_{2n}$ is a basis of $\mathfrak{g}_1$ and T a basis of $\mathfrak{g}_2$. Let p be an elliptic, homogeneous non-commuting polynomial in $2n$ variables, i.e. $p(\xi_1, \xi_2, \ldots, \xi_{2n}) \geq C\,|\xi|^d$, $C > 0$. An operator of the form $L = p(X_1, X_2, \ldots, X_{2n})$ will be said to be <u>homogeneous and elliptic in the generating directions</u>. It is known that L is analytic-hypoelliptic and C^∞ hypoelliptic if and only if the L^2 nullspace of L is nontrivial (see [10], [9], [7], [8], and [5]). The results announced here show that even if L is not hypoelliptic, it has a left inverse, modulo the projection onto its kernel, which preserves real analyticity, locally. More precisely, our main result is the following.

<u>Theorem 1.</u> <u>Let</u> L <u>be a homogeneous, left invariant differential operator on the Heisenberg group</u> G <u>elliptic in the generating directions. Then there are distributions</u> k_1 <u>and</u> k_2 <u>such that</u>

(1) $$Lf * k_1 = f - \Pi_1 f$$

(2) $$L(f * k_2) = f - \Pi_2 f$$

for $f \in C_0^\infty(G)$, where Π_1 and Π_2 are the orthogonal projections onto the L^2 nullspaces of L and its adjoint L^*, respectively, and $(*)$ denotes group convolution. Furthermore, the operators $f \to f * k_i$ and $f \to \Pi_i f$, $i = 1,2$, all preserve analyticity, locally.

Corollary. If u and f are smooth functions of compact support on G and

$$Lu = f \quad \text{in} \quad U, \tag{3}$$

where U is an open set, then $u_1 = (I - \Pi)\, u$ is analytic in every subset of U where f is, and u_1 also satisfies (3).

In the special case where $L = \Box_b^0$, the boundary Laplacian operator acting on 0-forms (see [2]) the analog of Theorem 1 was given by Greiner, Kohn, and Stein [4], who derived explicit formulas for k_i and Π_i. The analyticity of the projections Π_i was proved by Geller [3], who also proved the existence of distributions k_i, satisfying (1) and (2) and preserving local smoothness. The general result was conjectured by Stein [3]. See also Melin [6] for related results.

To prove Theorem 1, we use a standard reduction to the case where L is self adjoint and of high degree, in addition to satisfying the conditions of Theorem 1. The following is partly based on an idea of Beals and Greiner [1].

Theorem 2. Let L be a self-adjoint differential operator of high homogeneous degree d satisfying the conditions of Theorem 1. Then there is a closed contour Γ around 0 in $\mathbb{C}$ such that $L_\alpha = L - \alpha(-iT)^{d/2}$ is hypoelliptic for all $\alpha \in \Gamma$. There exist distributions k_α, $\alpha \in \Gamma$, *such that* $L_\alpha k_\alpha = \delta$ *and for any* $f \in C_0^\infty(G)$ *and any multi-index* β *the function* $\alpha \to ||D^\beta(f * k_\alpha)||_{L^\infty}$ *is bounded for* α *on* Γ. *Hence define* $K, S\colon C_0^\infty(G) \to C^\infty(G)$ *by*

$$Kf = \frac{1}{2\pi i}\int_\Gamma \alpha^{-1} f * k_\alpha d\alpha$$

and

$$Sf = \frac{1}{2\pi i}\int_\Gamma T^{d/2} f * k_\alpha d\alpha$$

Then

$$\text{(4)} \qquad LKf = K * Lf = f - Sf, \qquad f \in C_0^\infty(G)\ ,$$

and $S = \prod$, *the orthogonal projection onto the* L^2 *kernel of* L. *Furthermore,* K *and* S *preserve real analyticity, locally.*

The proof of Theorem 2 first requires constructing the k_α. For this we follow the method given by Métivier [7], checking that the k_α so obtained vary well with α. The first identity in (4) follows from the self adjointness of L, while the second is immediately obtained by writing $L = L_\alpha + \alpha(-iT)^{d/2}$. The proof that $S = \prod$ is accomplished by applying the irreducible unitary representations to both operators. Then the equality reduces to a resolvant identity, and the original identity follows by the Plancherel formula for G.

Finally, to show that K and S preserve real analyticity, it suffices to show that the operators $f \to f * k_\alpha$, each of which preserves real analyticity, satisfy estimates uniform in α for α on Γ. For this, we use the methods of the second author [9] to estimate the L^2 norms of derivatives of $f * k_\alpha$, checking again that the constants obtained may be chosen independent of α.

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