JOURNÉES ÉQUATIONS AUX DÉRIVÉES PARTIELLES

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Journées Équations aux dérivées partielles (1978), p. 1-2

<http://www.numdam.org/item?id=JEDP_1978____A3_0>

Conférence n° 3

ON SOME LOCALIZED ESTIMATES FOR PSEUDO-DIFFERENTIAL OPERATORS

by A. MELIN

Let $P = p^w(x,D)$ (Weyl convention) be a classical ps.d.op. in $\mathbb{R}^n$ with principal symbol p_m positively homogeneous of degree m. Let ρ_o, say $\rho_o = (0,\xi_o)$, be a point in the cotangent space of $\mathbb{R}^n$ and consider problems of the following types :

(A) Determine those μ for which there exist a constant C and a ps.d.op. R of order strictly less than μ near ρ_o such that
$\|u\|^2_{(\mu)} \le C\|Pu\|^2 + \|Ru\|^2$; $u \in C_o^\infty(\mathbb{R}^n)$,
or in case p is real, $p_m \ge 0$.

(B) Determine those μ for which there is a lower bound
$(Pu,u) \ge C\|u\|^2_{(\mu)} + \|Ru\|^2$.

Sometimes when μ is kept fixed we also look for the possible constants C that can occur.

In case $p_m(\rho_o) \neq 0$ the standard calculus for pseudo-differential operators gives us a simple answer. In the other cases one has to localize the estimates near the characteristic variety Σ and get corresponding problems for operators with polynomial coefficients obtained from the Taylor series of $(x,\xi) \to p(\rho + \lambda(x,\xi))$ when $\rho \in \Sigma$, and λ is a small parameter. Thus for example to have (B) with $\mu = (m-1)/2$ (the sharp Gårding inequality) and a constant C implies lower bounds for the eigenvalues of the harmonic oscillator type operators which are obtained from a Taylor expansion up to the second order along Σ. Sharper results in these direction are obtained by Hörmander [2].

In Egorov [1] it is shown that his theorem about the validity of (A) with $\mu = m - \delta$ under the condition (ψ) when not all the commutators p_I of length $|I|$ of Re p and Im p vanish when $|I|(1-\delta) \le 1$, essentially relies upon estimates of the following form :

$$M\|u\| + \|u_t'\| \le C_o \|u_t' - Q(t)u\| . \tag{1}$$

Here Q is either multiplication by a polynomial q(t) or an operator $v(y) \to [F(t,y) + G(t)D_y]\, v(y)$ with polynomial coefficients acting on $L^2(\mathbb{R}_y)$. The condition (ψ) implies that $(Q(t)v,v)$ can only change sign from - to + for fixed v and M is related to $\sum_{|I|(1-\delta)\le 1} |P_I|^{1/|I|}$. Thus for example $M = \Sigma |q^{(j)}(0)|^{1/(j+1)}$ in the first case. In this case a simple proof for (1) is given if one observes that there is a constant C_A only depending on A so that :

(2) $$\|u_t'\| + \|u\| \le C_A \|u' - hu\|$$

if h can only change sign from - to + and in addition satisfies the following :

(3) $$\text{measure } \{t; |h(t)| < A^{-1}\} < 1 ,$$

(4) $$\int_{\mathbb{R}} \max(0, -h'(t) - |h(t)|)dt \le A.$$

One then obtains (1) when $Q(t) = q(t)$ by a symplectic dilation.

References

[1] Yu. V. Egorov : Subelliptic operators . Usp. Mat. Nauk 30 : 2 (1975) and Usp. Mat. Nauk 30 : 3 (1975). See also Russian Math. Surveys 30 : 2 (1975) 59-118 and Russian Math Surveys 30 : 3 (1975) 55-105.

[2] L. Hörmander : General pseudodifferential operators. Lectures at Stanford University the Summer Quarter 1977.