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# Schrödinger operator with magnetic field in domain with corners

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## Abstract

We present here a simplified version of results obtained with F. Alouges, M. Dauge, B. Helffer and G. Vial (cf [4, 7, 9]). We analyze the Schrödinger operator with magnetic field in an infinite sector. This study allows to determine accurate approximation of the low-lying eigenpairs of the Schrödinger operator in domains with corners. We complete this analysis with numerical experiments.

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## 1. Introduction

Let  $\Omega \subset \mathbb{R}^2$  be an open, simply connected domain with Lipschitz boundary and let  $\nu$  be the unit outer normal on the boundary  $\Gamma = \partial\Omega$ . We assume that  $\nu$  is well defined on  $\Gamma$  with the possible exception of a finite number of points (the corners of  $\Omega$ ). We consider a cylindrical superconducting sample of cross section  $\Omega$  and we apply a constant magnetic field of intensity  $\sigma$  along the cylindrical axis. We denote by  $\kappa$  the characteristic of the sample, called the “Ginzburg-Landau parameter” and consider  $\kappa$  large (corresponding to a type II superconductor). Then, up to normalization factors, the free energy writes

$$\mathcal{G}(\psi, \mathcal{A}) = \frac{1}{2} \int_{\Omega} \left( |(\nabla - i\kappa\mathcal{A})\psi|^2 + \frac{\kappa^2}{2} (|\psi|^2 - 1)^2 + \kappa^2 |\operatorname{curl} \mathcal{A} - \sigma|^2 \right) dx. \quad (1)$$

The superconducting properties are described by the minimizers  $(\psi, \mathcal{A})$  of this Ginzburg-Landau functional  $\mathcal{G}$ . The complex-valued function  $\psi$  is the order parameter (cf [13]) ; the magnitude  $|\psi|^2$  represents the density of superconducting electrons and the phase determines the current flow. The vector field  $\mathcal{A}$  defined on  $\mathbb{R}^2$  is the magnetic potential and  $\mathcal{B} = \operatorname{curl} \mathcal{A}$  is the induced magnetic field. To determine the onset of the superconductivity, we linearize the Euler equation associated with (1)

near the normal state  $(\psi, \mathcal{A}) = (0, \sigma \mathcal{A}_0)$ , where

$$\mathcal{A}_0 := \frac{1}{2}(x_2, -x_1). \quad (2)$$

From now, we put  $\mathcal{A} = \mathcal{A}_0$  and assume  $\Omega$  bounded. Defining the change of parameter  $h = \frac{1}{\kappa\sigma}$ , we are interested in the asymptotic behavior, when  $h \rightarrow 0$ , of the Neumann realization  $P_h$  of the Schrödinger operator with a magnetic field and semi-classical parameter  $h > 0$ . We define the associated quadratic form  $p_h$  on  $H^1(\Omega)$  by :

$$p_h(u) = \int_{\Omega} |(h\nabla - i\mathcal{A})u(x)|^2 dx.$$

This leads to define the operator  $P_h = -(h\nabla - i\mathcal{A})^2$  on  $\mathcal{D}(P_h)$  with :

$$\mathcal{D}(P_h) = \left\{ u \in H^2(\Omega), \quad \nu \cdot (h\nabla - i\mathcal{A})u \Big|_{\Gamma} = 0 \right\}.$$

It is well known that the spectrum of the operator  $P_{h,\mathcal{A}}$  is invariant by gauge transformation. So, when  $\Omega$  is simply connected, the spectrum of  $P_h$  depends only on the magnetic field and not on the choice of the corresponding magnetic potential. Then, we denote by  $\mu_{h,n}$  the  $n$ -th eigenvalue of  $P_h$  for any  $\mathcal{A}'$  such that  $\text{curl } \mathcal{A}' = \text{curl } \mathcal{A}$ .

Many papers have been devoted to this problem among which we quote the works by Bernoff-Sternberg [3], Lu-Pan [19, 20], Helffer-Mohamed [15, 16] and more recently Fournais-Helffer [12]. These papers deal with the case of regular domains and propose an asymptotics of the bottom of the spectrum. Although the interest for a non smooth domain is often mentioned in the physical literature, there are very few mathematical papers : We only know the contributions by Pan [22] and Jadallah [18] which deal with very particular domains like a square or a quarter plane. Our goal is to give asymptotics for the low-lying eigenvalues and localization of the corresponding eigenvectors in a domain with corners.

In the analysis of smooth domains, the model operator  $-(\nabla - i\mathcal{A})^2$  on  $\mathbb{R}^2$  and  $\mathbb{R} \times \mathbb{R}^+$  was playing an important role. Our new model for domains with corners is the operator  $-(\nabla - i\mathcal{A})^2$  in angular sectors. We analyze this model in Section 2. We recall results proved in [7]. We use these results, in Section 3, to construct quasi-modes for the operator  $P_h$  in polygonal domain. This gives the asymptotics of  $\mu_{h,n}$  when the domain is a polygon in Section 4. We propose also some numerical experiments which show the decay of the eigenfunctions, the convergence of the eigenvalues and a numerical illustration of a tunnelling effect.

The bottom of the spectrum of the Schrödinger operator with magnetic field on an infinite sector is an eigenvalue when the opening is less than  $\pi/2$  whereas it equals the bottom of the essential spectrum for the half-plane. This generates very different results for a polygonal domain and for a smooth domain.

## 2. The model case of an infinite sector

This section is devoted to the analysis of the Neumann realization of the Schrödinger operator with magnetic field  $-(\nabla - i\mathcal{A})^2$  on an infinite sector. We recall results developed in [5, 7] and we just propose some sketches of proofs here.

## 2.1. Notations

Let  $G^\alpha$  be the sector in  $\mathbb{R}^2$  with opening  $\alpha$  and  $\mathbf{X} = (X_1, X_2)$  the coordinates on the sector. The Neumann realization  $Q^\alpha$  of the Schrödinger operator with magnetic field  $-(\nabla - i\mathcal{A})^2$  on  $G^\alpha$  is associated with the quadratic form  $q^\alpha$  defined on the variational space  $\mathcal{V}(q^\alpha)$  as follows :

$$\mathcal{V}(q^\alpha) = \left\{ \Psi \in L^2(G^\alpha), (\nabla - i\mathcal{A})\Psi \in L^2(G^\alpha) \right\}, \quad (3)$$

$$q^\alpha(\Psi) = \int_{G^\alpha} |(\nabla - i\mathcal{A})\Psi(\mathbf{X})|^2 d\mathbf{X}, \quad \forall \Psi \in \mathcal{V}(q^\alpha). \quad (4)$$

The norm attached with the space  $\mathcal{V}(q^\alpha)$  is

$$\|\Psi\|_{\mathcal{V}(q^\alpha)}^2 = \|\Psi\|_{L^2(G^\alpha)}^2 + \|(\nabla - i\mathcal{A})\Psi\|_{L^2(G^\alpha)}^2.$$

Note that if  $\Psi \in \mathcal{V}(G^\alpha)$ , then for any ball  $B$ ,  $\Psi \in H^1(G^\alpha \cap B)$ . We denote by  $Q^\alpha$  the operator associated with the form  $q^\alpha$ . Then,

$$Q^\alpha = -(\nabla - i\mathcal{A})^2,$$

is defined on  $\mathcal{D}(Q^\alpha)$  with :

$$\mathcal{D}(Q^\alpha) = \left\{ \Psi \in \mathcal{V}(q^\alpha), (\nabla - i\mathcal{A})^2\Psi \in L^2(G^\alpha), \nu \cdot (\nabla - i\mathcal{A})\Psi \Big|_{\partial G^\alpha} = 0 \right\}.$$

**Definition 2.1.** Let  $\mu_k(\alpha)$  be the  $k$ -th smallest element of the spectrum of  $Q^\alpha$ , given by the max-min principle:

$$\mu_k(\alpha) = \max_{\Psi_1, \dots, \Psi_{k-1}} \min \left\{ \frac{q^\alpha(\Psi)}{\langle \Psi, \Psi \rangle}, \Psi \in \mathcal{V}(q^\alpha), \Psi \in [\Psi_1, \dots, \Psi_{k-1}]^\perp \right\}. \quad (5)$$

Here  $\langle \cdot, \cdot \rangle$  denotes the hermitian scalar product of  $L^2(G^\alpha)$ .

## 2.2. Essential spectrum

**Proposition 2.2.** The infimum of the essential spectrum of  $Q^\alpha$  is equal to  $\Theta_0 := \mu_1(\pi)$ .

*Proof.* This result is a consequence of the Persson Lemma (cf [23]) which can be generalized to unbounded domains of  $\mathbb{R}^2$  and Neumann realizations :

**Lemma 2.3.** Let  $\Omega$  be an unbounded domain of  $\mathbb{R}^2$  with Lipschitz boundary. We denote by  $\inf \sigma_{\text{ess}}(-(\nabla - i\mathcal{A})^2)$  the bottom of the essential spectrum, then :

$$\inf \sigma_{\text{ess}}(-(\nabla - i\mathcal{A})^2) = \lim_{r \rightarrow \infty} \Sigma_r(-(\nabla - i\mathcal{A})^2), \quad (6)$$

with, denoting  $\Omega_r = \{\mathbf{X} \in \bar{\Omega} \mid |\mathbf{X}| > r\}$  :

$$\Sigma_r(-(\nabla - i\mathcal{A})^2) := \inf_{\phi \in C_0^\infty(\Omega_r), \phi \neq 0} \frac{\int_{\Omega} |(\nabla - i\mathcal{A})\phi(\mathbf{X})|^2 d\mathbf{X}}{\int_{\Omega} |\phi(\mathbf{X})|^2 d\mathbf{X}}. \quad (7)$$

Relying on this lemma, we use a partition of unity which splits the sector into three subdomains which can be compared to the models  $\mathbb{R}^2$  or  $\mathbb{R} \times \mathbb{R}^+$  respectively.  $\square$

## 2.3. Decay of eigenfunctions

**Proposition 2.4.** *Let  $k$  be a positive integer and  $\alpha > 0$  such that  $\mu_k(\alpha) < \Theta_0$ . We denote by  $\Psi_k^\alpha$  a normalized eigenfunction associated with  $\mu_k(\alpha)$ . Then  $\Psi_k^\alpha$  satisfies the following exponential decay estimate*

$$\forall \varepsilon > 0, \exists C_{\varepsilon, \alpha} : \left\| e^{(\sqrt{\Theta_0 - \mu_k(\alpha) - \varepsilon})|\mathbf{X}|} \Psi_k^\alpha \right\|_{\mathcal{V}(G^\alpha)} \leq C_{\varepsilon, \alpha}. \quad (8)$$

*Proof.* Agmon's estimates (cf [1]) are useful to prove this result. We recall their principle. Let  $\phi$  be a uniformly lipschitzian function on  $G^\alpha$ , then, by assumption on  $\Psi_k^\alpha$  :

$$\int_{G^\alpha} (\mu_k(\alpha) + |\nabla \phi|^2) e^{2\phi} |\Psi_k^\alpha|^2 d\mathbf{X} = \int_{G^\alpha} |(\nabla - i\mathcal{A})(e^\phi \Psi_k^\alpha)|^2 d\mathbf{X}. \quad (9)$$

Let  $B_R$  be the ball in  $\mathbb{R}^2$  centered on 0 with radius  $R$  and  $\chi_1, \chi_2$  be real, positive, smooth functions, with support respectively in  $B_2$  and  $\mathbb{R}^2 \setminus B_1$ , and such that  $|\chi_1|^2 + |\chi_2|^2 \equiv 1$ . We define  $\chi_j^R := \chi_j(\frac{\cdot}{R})$ , then :

$$q_\alpha(e^\phi \Psi_k^\alpha) = \sum_{j=1}^2 q_\alpha(\chi_j^R e^\phi \Psi_k^\alpha) - \frac{1}{R^2} \sum_{j=1}^2 \|e^\phi \Psi_k^\alpha |\nabla \chi_j^R|\|_{L^2(G^\alpha)}^2.$$

The two last relations combined with the positivity of  $q_\alpha(\chi_1^R e^\phi \Psi_k^\alpha)$  lead to :

$$q_\alpha(\chi_2^R e^\phi \Psi_k^\alpha) \leq \int_{G^\alpha} \left( \mu_k(\alpha) + |\nabla \phi|^2 + \frac{C}{R^2} \right) e^{2\phi} |\Psi_k^\alpha|^2 d\mathbf{X}. \quad (10)$$

We can prove (cf Lemma 2.3 and [7] for details) that :

$$q_\alpha(\chi_2^R e^\phi \Psi_k^\alpha) \geq \left( \Theta_0 - \frac{C}{R^2} \right) \|\chi_2^R e^\phi \Psi_k^\alpha\|_{L^2(G^\alpha)}^2. \quad (11)$$

To obtain a  $L^2$ -estimate, we put together (10) and (11). We bound  $\|\chi_2^R e^\phi \Psi_k^\alpha\|_{L^2(G^\alpha)}^2$  from below by  $\|e^\phi \Psi_k^\alpha\|_{L^2(G^\alpha \setminus B_{2R})}^2$ , choose  $\phi(x) := \sqrt{\Theta_0 - \mu_k(\alpha) - \varepsilon} |\mathbf{X}|$  and split the integral over  $G^\alpha$  in two parts, respectively over  $G^\alpha \setminus B_{2R}$  and  $B_{2R} \cap G^\alpha$ . To end the proof, it is enough to use (9) again.  $\square$

## 2.4. Estimates of $\mu_1(\alpha)$

**Theorem 2.5.**

- (i) For all  $\alpha \in (0, \frac{\pi}{2}]$ ,  $\mu_1(\alpha) < \Theta_0$  and, therefore,  $\mu_1(\alpha)$  is an eigenvalue.
- (ii) There exists a real sequence  $(m_j)_{j \in \mathbb{N}}$ , with  $m_0 = 1/\sqrt{3}$ , such that :

$$\mu_1(\alpha) \sim \alpha \sum_{j=0}^{\infty} m_j \alpha^{2j} \text{ as } \alpha \rightarrow 0.$$

Furthermore,  $\mu_1(\alpha) \leq \alpha/\sqrt{3}$  for any  $\alpha \in (0, \pi]$ .

*Proof.* After a change of variables, a scaling and a gauge transformation, we get a new operator which is now defined on a constant domain  $\omega: = \mathbb{R}^+ \times ]-\frac{1}{2}, \frac{1}{2}[$  with coordinates  $(t, \eta)$  (instead of the constant operator  $-(\nabla - i\mathcal{A})^2$  on an  $\alpha$ -dependent domain  $G^\alpha$ ). This new operator is associated with the sesquilinear form :

$$a_\alpha(u, v): = \int_\omega \left( 2\alpha t(D_t - \eta)u \overline{(D_t - \eta)v} + \frac{1}{2\alpha t} D_\eta u \overline{D_\eta v} \right) dt d\eta,$$

defined on :

$$\mathcal{V}: = \left\{ u \in L^2(\omega) \mid \sqrt{t}(D_t - \eta)u \in L^2(\omega), \frac{1}{\sqrt{t}}D_\eta u \in L^2(\omega) \right\},$$

We look for two sequences :  $(u_k)_{k \in \mathbb{N}}$  and  $(m_k)_{k \in \mathbb{N}}$  with  $m_k$  real such that for all  $n \in \mathbb{N}^*$ , if we define  $U^{(n)} = \sum_{k=0}^n \alpha^{2k} u_k$  and  $\mu^{(n)}(\alpha) = \sum_{k=0}^n \alpha^{2k} m_k$ , then, modulo  $\mathcal{O}_n(\alpha^{2n+2})$ , we have :

$$a_\alpha(U^{(n)}, v) \equiv \mu^{(n)}(\alpha) \langle U^{(n)}, v \rangle_{L^2(\omega)}, \quad \forall v \in \mathcal{V}.$$

We expand the equation in powers of  $\alpha$  and express that the coefficients of  $\alpha^{2k}$  ( $k \geq -1$ ) should cancel. The construction shows that functions  $u_k$  belong to the space of polynomial functions in  $\eta$  whose coefficients are in  $\mathcal{S}(\overline{\mathbb{R}^+})$ .  $\square$

**Remark 2.6.** From the expression of the form  $a_\alpha$ , we immediately see that the function  $\alpha \mapsto \alpha\mu_1(\alpha)$  is increasing and  $\alpha \mapsto \mu_1(\alpha)/\alpha$  is decreasing over  $(0, 2\pi)$ . According to results on  $\mu_1(\alpha)$  (cf [7]) and numerical computations (cf [4]), we conjecture that  $\mu_1$  is strictly increasing from  $(0, \pi]$  onto  $(0, \Theta_0]$  and equal to  $\Theta_0$  on  $[\pi, 2\pi)$ . Figure 1 presents numerical estimates of  $\mu_1(\alpha)$  that we have obtained (with G. Vial) using a finite element method. To realize these computations, we consider a truncated sector  $G_c^\alpha$ . We keep Neumann magnetic boundary conditions on the common boundary between  $G_c^\alpha$  and  $G^\alpha$  and a Dirichlet conditions on the other part of the boundary of  $G_c^\alpha$ . So we are ensured to obtain an upper-bound of  $\mu_1(\alpha)$ ,  $\alpha \in \{k\pi/100, k = 1, \dots, 85\}$ . For any opening, we consider a uniform mesh with 48 quadrangular elements and a tensor product polynomial of degree 10.

### 3. Construction of quasi-modes for polygonal domains

Results presented in this section are proved in [9].

#### 3.1. Notation and localized model operators

Let  $\Omega \in \mathbb{R}^2$  be a convex bounded polygon with straight edges,  $\Sigma$  be the set of its vertices  $\mathbf{s}$ , and  $\alpha_s$  be its angle at  $\mathbf{s} \in \Sigma$ . The spectrum of  $P_h$  is in close relation with the spectra of the model operators  $Q^{\alpha_s}$ , for  $\mathbf{s}$  describing the set of corners  $\Sigma$ . As a first step in the explanation of this relation, we introduce, for each vertex  $\mathbf{s}$ , the semi-model operator  $\tilde{Q}_{h,\mathbf{s}}$  defined by the same operator as  $P_h$ , but on the infinite plane sector  $\tilde{G}_\mathbf{s}$  which coincides with  $\Omega$  near the vertex  $\mathbf{s}$ : Let  $d = \min_{\mathbf{s}' \in \Sigma} d(\mathbf{s}, \mathbf{s}')$

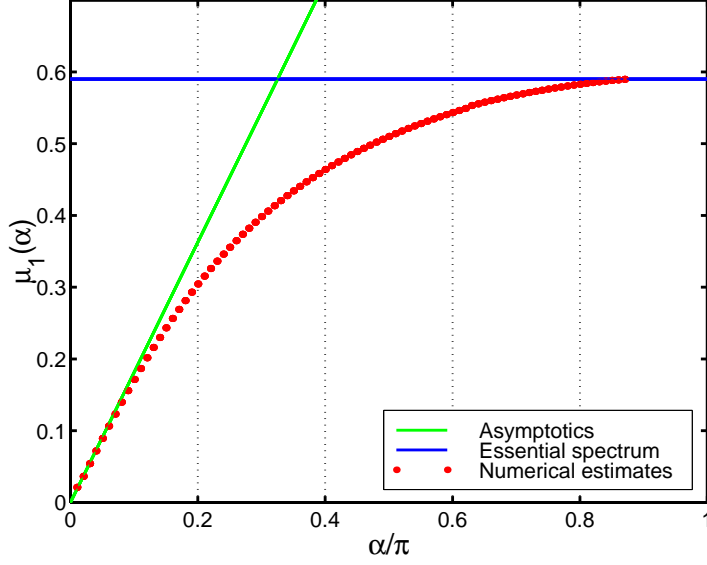


Figure 1: Estimates of  $\mu_1(\alpha)$  by a finite element method.

and  $\mathcal{R}_s$  be the rotation of angle  $\beta_s$  such that

$$\{\mathcal{R}_s(x - s), x \in \Omega \cap \mathcal{B}(s, d)\} \subset G^{\alpha_s},$$

and let us denote

$$\tilde{G}_s = \{\mathcal{R}_s^{-1}X + s, X \in G^{\alpha_s}\}.$$

Finally  $\tilde{Q}_{h,s}$  is the operator  $-(h\nabla - i\mathcal{A})^2$  on  $\tilde{G}_s$ .

**Lemma 3.1.** *The following relation holds between the spectra of the operators  $\tilde{Q}_{h,s}$  and  $Q^{\alpha_s}$  respectively denoted by  $\sigma(\tilde{Q}_{h,s})$  and  $\sigma(Q^{\alpha_s})$  :*

$$\sigma(\tilde{Q}_{h,s}) = h \sigma(Q^{\alpha_s}).$$

*The corresponding eigenvectors are deduced from each other by a change of variables and a gauge transformation*

$$\begin{aligned} \mathcal{V}(q^{\alpha_s}) &\longrightarrow \mathcal{V}(\tilde{q}_{h,s}) = \left\{ \psi \in L^2(\tilde{G}_s), (h\nabla - i\mathcal{A})\psi \in L^2(\tilde{G}_s) \right\}, \\ \Psi_k^\alpha &\longmapsto \tilde{\psi}_{h,s,k} \quad \text{s.t.} \quad \tilde{\psi}_{h,s,k}(x) = \frac{1}{\sqrt{h}} \exp\left(\frac{i}{2h}x \wedge s\right) \Psi_k^\alpha\left(\frac{\mathcal{R}_s(x-s)}{\sqrt{h}}\right). \end{aligned} \quad (12)$$

We now use results on angular sectors and Lemma 12 to construct functions which are good approximations of the eigenfunctions of  $P_h$ .

### 3.2. Quasi-modes

**Lemma 3.2.** *Let  $s \in \Sigma$  and  $d_s$  be the distance to other vertices*

$$d_s = \text{dist}(s, \Sigma \setminus \{s\}).$$

Let  $d' < d_s$  and  $\chi_s$  be a smooth cut-off function with support in  $\mathcal{B}(s, d_s)$ , equal to 1 in  $\mathcal{B}(s, d')$  and such that  $0 \leq \chi_s \leq 1$ .

We consider  $\Psi^{\alpha_s}$  a normalized eigenfunction of  $Q^{\alpha_s}$  on  $G^{\alpha_s}$  for the eigenvalue  $\mu_k(\alpha_s)$ . Using the transformation (12), we define :

$$\tilde{\psi}_{h,s,k}(x) = \frac{e^{\frac{i}{2h}x \wedge s}}{\sqrt{h}} \Psi_k^{\alpha_s} \left( \frac{\mathcal{R}_s(x - s)}{\sqrt{h}} \right) \text{ on } \tilde{G}_s, \quad (13)$$

$$\psi_{h,s,k}(x) = \chi_s(x) \tilde{\psi}_{h,s,k}(x) \text{ on } \Omega. \quad (14)$$

Then, for any  $\varepsilon > 0$ , there exists  $C_\varepsilon$  such that

$$\begin{aligned} |1 - \|\psi_{h,s,k}\|_\Omega^2| &\leq C_\varepsilon \exp\left(-\frac{2}{\sqrt{h}} \left(d' \sqrt{\Theta_0 - \mu_k(\alpha_s)} - \varepsilon\right)\right), \\ \left| \frac{p_h(\psi_{h,s,k})}{\|\psi_{h,s,k}\|_\Omega^2} - h\mu_k(\alpha_s) \right| &\leq C_\varepsilon \exp\left(-\frac{2}{\sqrt{h}} \left(d' \sqrt{\Theta_0 - \mu_k(\alpha_s)} - \varepsilon\right)\right), \\ \|P_h \psi_{h,s,k} - h\mu_k(\alpha_s) \psi_{h,s,k}\|_\Omega &\leq C_\varepsilon \exp\left(-\frac{1}{\sqrt{h}} \left(d' \sqrt{\Theta_0 - \mu_k(\alpha_s)} - \varepsilon\right)\right), \end{aligned}$$

where  $\|\cdot\|_\Omega$  denotes the  $L^2$ -norm on  $\Omega$ .

The proof relies on the decay of the eigenfunctions  $\Psi_k^{\alpha_s}$ . The following lemma shows how we can split the corners of the polygon to obtain global informations.

### 3.3. Partition of unity

**Lemma 3.3.** For any  $s \in \Sigma$ , we consider a real-valued cut-off function  $\chi_s$  with support in  $\mathcal{B}(s, d_s)$ . We assume furthermore that for any  $s \neq s'$ ,  $\text{supp}\chi_s \cap \text{supp}\chi_{s'} = \emptyset$ . We define  $\chi_0$  on  $\Omega$  by  $\chi_0^2 = 1 - \sum_{s \in \Sigma} \chi_s^2$ . Then, for any  $\tilde{\psi} \in H^1(\Omega)$ ,

$$p_h(\tilde{\psi}) = \sum_{s \in \Sigma \cup \{0\}} p_h(\chi_s \tilde{\psi}) - h^2 \sum_{s \in \Sigma \cup \{0\}} \|\tilde{\psi} \nabla \chi_s\|_{L^2(\Omega)}^2.$$

## 4. Approximation of $\sigma(P_h)$ with the model operators

### 4.1. Asymptotics

**Notation 4.1.** Let us denote by  $\lambda_n$  the  $n$ -th eigenvalue of  $\oplus_{s \in \Sigma} Q^{\alpha_s}$  counted with multiplicity as defined by the min-max principle, and let  $N$  be the largest integer such that  $\lambda_N < \Theta_0$ . We assume that  $N \geq 1$ . For any  $n \leq N$ , we denote by  $\Sigma_n$  the subset of vertices

$$\Sigma_n = \left\{ s \in \Sigma, \lambda_n \text{ is an eigenvalue for } Q^{\alpha_s} \right\},$$

and by  $r_n$  the distance

$$r_n = r(\lambda_n) = \min_{s \in \Sigma_n} d(s, \Sigma \setminus \{s\}).$$



Let  $n \leq N$ . We denote by  $\mu_{h,n}$  the  $n$ -th eigenvalue of  $P_h$  counted with multiplicity. For any  $\mathbf{s} \in \Sigma_n$ , we denote by  $\Psi^{\alpha_{\mathbf{s}}}$  a normalized eigenvector for  $Q^{\alpha_{\mathbf{s}}}$  associated with  $\lambda_n$  and by  $\tilde{\psi}_{h,\mathbf{s}}$  the function deduced from  $\Psi^{\alpha_{\mathbf{s}}}$  by (13), then  $\psi_{h,\mathbf{s}}$  is a normalized eigenfunction of  $\tilde{Q}_{h,\mathbf{s}}$  for  $h\lambda_n$ . Let  $\varepsilon > 0$ . We consider a smooth cut-off function  $\chi_{\mathbf{s}} \in C_0^\infty(\Omega, [0, 1])$  as in Lemma 3.2 with  $d' < r_n$  and we define  $\psi_{h,\mathbf{s}} = \chi_{\mathbf{s}}\tilde{\psi}_{h,\mathbf{s}}$  as in (14).

As we will see in the following theorem, according to repetitions of the same values in  $\{\lambda_1, \dots, \lambda_N\}$ , the eigenvalues  $\mu_{h,n}$  are gathered into clusters.

**Theorem 4.2.** *With Notation 4.1, for any  $\varepsilon > 0$ , there exists  $C_\varepsilon$  such that for any  $n \leq N$ ,*

$$\begin{aligned} \mu_{h,1} &\leq h\lambda_1 + C_\varepsilon \exp\left(-\frac{2}{\sqrt{h}}\left(r_1\sqrt{\Theta_0 - \lambda_1} - \varepsilon\right)\right), \\ |\mu_{h,n} - h\lambda_n| &\leq C_\varepsilon \exp\left(-\frac{1}{\sqrt{h}}\left(r_n\sqrt{\Theta_0 - \lambda_n} - \varepsilon\right)\right). \end{aligned}$$

*Proof.* Upper-bound of  $\mu_{h,1}$  is a consequence of Lemma 3.2 applied with  $\mu_k(\alpha_{\mathbf{s}}) = \lambda_1$  and  $d' = r_1 - \varepsilon$  and the min-max principle.

Let  $n \leq N$ ,  $\mathbf{s} \in \Sigma_n$  and  $d' = r_n - \varepsilon$ . We deduce from Lemma 3.2 and the spectral theorem that :

$$d(\sigma(P_h), h\lambda_n) \leq C_\varepsilon \exp\left(-\frac{1}{\sqrt{h}}\left(r_n\sqrt{\Theta_0 - \lambda_n} - \varepsilon\right)\right). \quad (15)$$

To prove a lower bound of  $\mu_{h,n}$ , we use [10, 24]. □

To approximate eigenvectors of  $P_h$ , we have to take account of clusters as explained in the following result.

**Notation 4.3.** Using Notations 4.1, we denote by  $\{\Lambda_1 < \dots < \Lambda_M\}$  the set of distinct values in  $\{\lambda_1, \dots, \lambda_N\}$ . Let  $m \leq M$ , we define the distances

$$\rho_m = r(\Lambda_m).$$

For  $n \leq N$ , we denote by  $(\mu_{h,n}, u_{h,n})$  the  $n$ -th eigenpair of  $P_h$ . We introduce the cluster of eigenspaces of  $P_h$  by

$$F_{h,m} = \text{span}\{u_{h,n} \text{ for any } n \text{ such that } \lambda_n = \Lambda_m\},$$

and the cluster of quasi-modes

$$E_{h,m} = \text{span}\{\psi_{h,\mathbf{s},k} = \chi_{\mathbf{s}}\tilde{\psi}_{h,\mathbf{s},k} \text{ for any } \mathbf{s} \in \Sigma, k \geq 1 \text{ such that } \mu_k(\alpha_{\mathbf{s}}) = \Lambda_m\}.$$

Here,  $\chi_{\mathbf{s}}$  is a real-valued smooth cut-off function equal to 1 in  $\mathcal{B}(\mathbf{s}, \rho_m - \delta)$ .

**Theorem 4.4.** *Under Notation 4.1 and 4.3, for any  $\varepsilon > 0$ , there exists  $C_\varepsilon$  such that for any  $m \leq M$ ,*

$$d(E_{h,m}; F_{h,m}) \leq C_\varepsilon \exp\left(-\frac{1}{\sqrt{h}}\left((\rho_m - \delta)\sqrt{\Theta_0 - \Lambda_m} - \varepsilon\right)\right),$$

where, if we denote  $\Pi_{E_{h,m}}, \Pi_{F_{h,m}}$  the orthogonal projections on  $E_{h,m}$  and  $F_{h,m}$  respectively,  $d$  is the distance defined by :

$$d(E_{h,m}; F_{h,m}) = \|\Pi_{E_{h,m}} - \Pi_{F_{h,m}}\Pi_{E_{h,m}}\|_{L^2(\Omega)}.$$

*Proof.* The proof of this theorem relies on Proposition 4.1.1, p. 30 of [14] (cf also [17]) which we recall now :

**Theorem 4.5.** *Let  $A$  be a self-adjoint operator in a Hilbert space  $\mathcal{H}$ . Let  $I \subset \mathbb{R}$  be a compact interval,  $\psi_1, \dots, \psi_N \in \mathcal{H}$  linearly independents in  $\mathcal{D}(A)$  and  $\mu_1, \dots, \mu_N$  such that :*

$$A\psi_j = \mu_j\psi_j + r_j \text{ with } \|r_j\|_{\mathcal{H}} \leq \eta. \quad (16)$$

*Let  $a > 0$  and assume that  $\sigma(A) \cap (I + \mathcal{B}(0, 2a) \setminus I) = \emptyset$ . Then, if  $E$  is the space spanned by  $\psi_1, \dots, \psi_N$  and if  $F$  is the space associated with  $\sigma(A) \cap I$ , we have :*

$$d(E, F) \leq \frac{\eta\sqrt{N}}{a\sqrt{\lambda_S^{\min}}}, \quad (17)$$

*where  $\lambda_S^{\min}$  is the smallest eigenvalue of  $S = (\langle \psi_j, \psi_k \rangle_{\mathcal{H}})$  and  $d$  is the non-symmetric distance as :*

$$d(E, F) = \|\Pi_E - \Pi_F\Pi_E\|_{\mathcal{H}},$$

*denoting by  $\Pi_E, \Pi_F$  the orthogonal projections on  $E$  and  $F$  respectively.*

□

## 4.2. Numerical experiments

This section is devoted to numerical computations on a square. In this case, we have

$$\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 \approx 0.509905.$$

We present computations of the eigenpairs realized by M. Dauge with the Finite Element Method code MELINA (cf [21]). For any  $h$ , we use a uniform mesh with 64 elements and tensor product polynomial of degree 10. When  $h$  is very small, very fast oscillations (cf Figure 4) appear on the eigenfunctions as expected in (12) and it is better to increase the degree of the polynomial than reduce the size of the mesh. We can note this by looking at [8, 2, 6] where we use an adapted mesh refinement based on a posteriori error estimates and a low order degree; we need many elements.

Figure 2 presents the behavior of the twelve first eigenvalues  $\mu_{h,n}/h$  for  $n = 1, \dots, 12$ , as function of  $1/h$ . We draw the exponential tube

$$\lambda_1 \pm \exp\left(-\frac{r_1\sqrt{\Theta_0 - \lambda_1}}{\sqrt{h}}\right).$$

According to Theorem 4.2, we expect that the four first eigenvalues are concentrated in this tube.

Let us now present the modulus and the real part of the first eigenfunction according  $h$  in Figures 3 and 4 respectively. We can observe the exponential concentration near the corners in  $\sqrt{h}$ . Figure 4 displays concentration in  $e^{-c/\sqrt{h}}$  together with oscillations in  $1/h$  which appear in the construction of the quasi-modes in (13). Looking at the symmetry of these figures, we see the linear combination of the four modes constructed from the quarter plane. Figure 4 is a numerical illustration of a tunnelling effect. We hope to propose a theoretical interpretation by analyzing an interaction matrix.

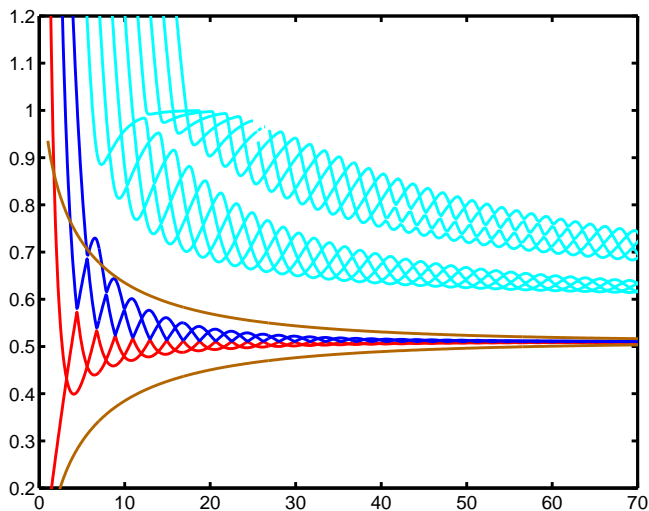


Figure 2:  $h^{-1}\mu_{h,n}$  versus  $h^{-1}$

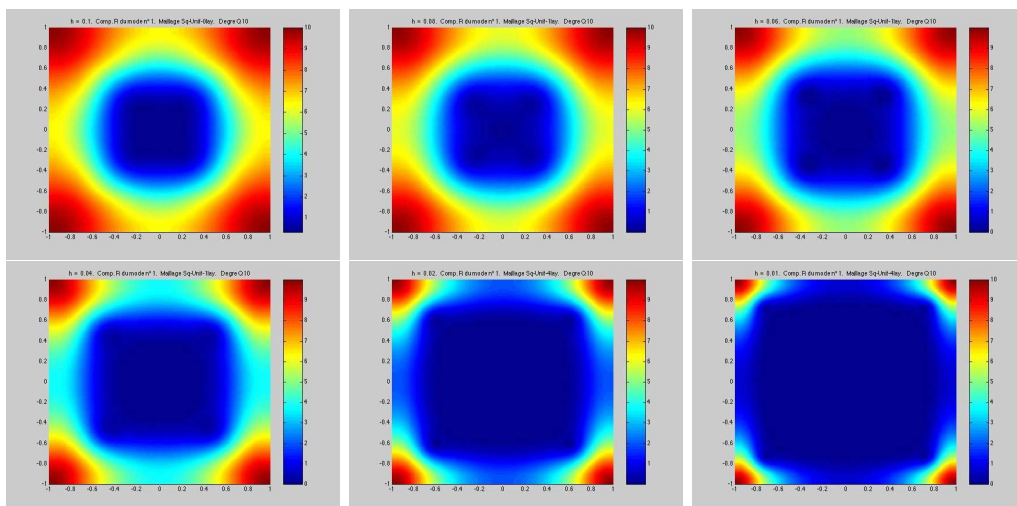


Figure 3:  $h = 0.1, 0.08, 0.06, 0.04, 0.02, 0.01$

## 5. Conclusion

Even if we have given some informations about the bottom of the spectrum of  $Q^\alpha$  as a function of  $\alpha$ , an open problem is to prove the monotonicity of  $\mu_1(\alpha)$ .

Furthermore, this paper completes the results of Helffer-Morame [16], Jadallah [18], Pan [22] by dealing with the low-lying eigenstates of the Schrödinger operator with constant magnetic field in a polygon and proving the localization of the eigenfunctions. We can generalize these results (with some assumptions, cf [9]) to the Schrödinger operator with non constant magnetic field in a bounded open domain with a curvilinear boundary.

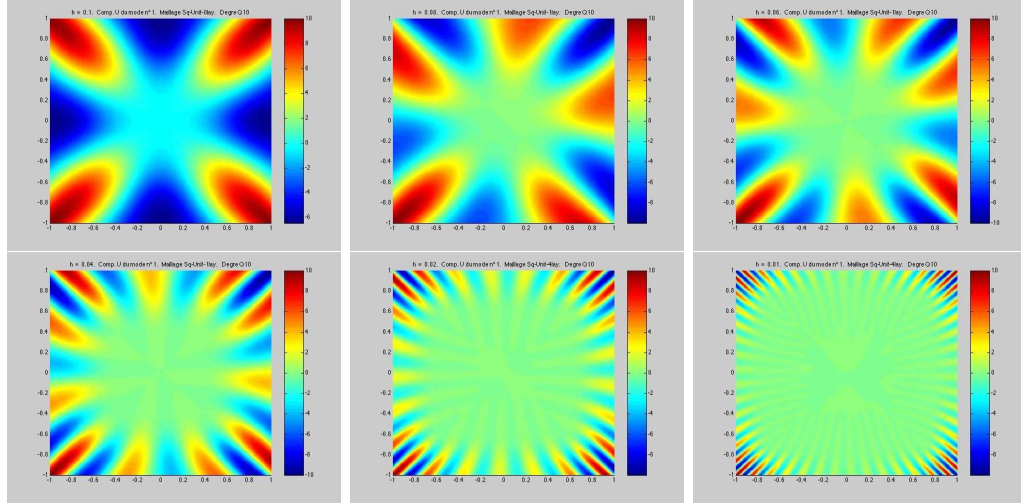


Figure 4:  $h = 0.1, 0.08, 0.06, 0.04, 0.02, 0.01$

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