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Global well-posedness and scattering for the mass-critical NLS


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Suppose \( u(t, x) \) is a solution to the nonlinear partial differential equation

\[
iu_t + \Delta u = \mu |u|^{4/d}u,
\]

\( u(0, x) = u_0 \in L^2(\mathbb{R}^d), \)

\( \mu = \pm 1, \mu = +1 \) refers to the defocusing case and \( \mu = -1 \) refers to the focusing case.

**Definition 0.1.** (0.1) is said to be globally well-posed if a solution \( u(t, x) \) to (0.1) exists for all time,

\[
u(t, x) \in C^0_t(\mathbb{R}; L^2(\mathbb{R}^d)) \cap L^{2(d+2)}_{t,loc}(\mathbb{R}; L^{2(d+2)}(\mathbb{R}^d)),
\]

and a solution to (0.1) depends continuously on \( u_0 \) in the \( L^2(\mathbb{R}^d) \) topology.

**Definition 0.2.** A global solution to (0.1) is said to scatter if there exist \( u_\pm \in L^2(\mathbb{R}^d) \) such that

\[
\|u(t, x) - e^{it\Delta}u_\pm\|_{L^2(\mathbb{R}^d)} \to 0,
\]

as \( t \to +\infty \) and

\[
\|u(t, x) - e^{it\Delta}u_-\|_{L^2(\mathbb{R}^d)} \to 0
\]

as \( t \to -\infty \). Additionally we say a solution to (0.1) scatters forward in time if it satisfies (0.3) and backward in time if it satisfies (0.4).

The first progress toward proving well-posedness of (0.1) was

**Theorem 0.1.** (0.1) is locally well-posed on \([-T, T]\) for some \( T(\|u_0\|_{H^1(\mathbb{R}^d)}) > 0. \)

**Proof:** See [6]. □

Furthermore, it is possible to use conserved quantities of (0.1) to upgrade theorem 0.1 to global well-posedness. (0.1) has the conserved quantities mass,

\[
M(u(t)) = \int |u(t, x)|^2 dx = M(u(0)),
\]

and energy

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In the defocusing case (0.6) is positively definite, which implies \( \|u(t)\|_{H^1(\mathbb{R}^d)} \) is uniformly bounded by \( E(u(0)) \) which is finite by the Sobolev embedding theorem. By (0.1) (0.1) is globally well-posed for \( u_0 \in H^1(\mathbb{R}^d), \mu = +1 \).

In the focusing case (0.6) is not positive definite. Therefore having \( E(u(0)) \) finite is not enough to prove global well-posedness because \( \|u(t)\|_{H^1(\mathbb{R}^d)} \) and \( \|u(t)\|_{L^{2(2d+2)}}(\mathbb{R}^d) \) can and do blow up at the same rate, precisely canceling to maintain conservation of energy.

For \( \|u(t)\|_{L^2(\mathbb{R}^d)} \) below a certain threshold it is still possible to prove global well-posedness and scattering in the case when \( \mu = -1 \) using the Gagliardo-Nirenberg inequality.

**Theorem 0.2.** If \( Q \) is the positive solution to the elliptic partial differential equation

\[
\Delta Q + Q^{1+4/d} = Q, \tag{0.7}
\]

the Sobolev embedding theorem has the best constant

\[
\|u\|_{L^{2(2d+2)}}(\mathbb{R}^d)^{2/4/d} \leq \|Q\|_{L^2(\mathbb{R}^d)^{2}} \|\nabla u\|_{L^2(\mathbb{R}^d)^{2/d}}. \tag{0.8}
\]

*Proof:* See [30], [43], [44], and [5]. □

Combining theorem 0.2 with (0.5) proves (0.1) when \( \|u_0\|_{L^2(\mathbb{R}^d)} < \|Q\|_{L^2(\mathbb{R}^d)} \). Furthermore, by (0.7)

\[
 u(t, x) = e^{it}Q(x) \tag{0.9}
\]

is a solution to (0.1) when \( \mu = -1 \). This is a solution that certainly fails to scatter. Applying the conformal symmetry

**Theorem 0.3.** \( u \) is a solution to (0.1) if and only if

\[
v(t, x) = \frac{1}{|t|^{d/2}} u\left(-\frac{1}{t}, \frac{x}{t}\right) e^{i|x|^2/t} \tag{0.10}
\]

solves (0.1).

We obtain a solution to (0.1) that fails to be globally well-posed.
twice. Since (0.11) is positive definite, this implies a solution to (0.1) can only exist in both directions for finite time. Such solutions are relatively straightforward to construct when \( \|u_0\|_{L^2(\mathbb{R}^d)} > \|Q\|_{L^2(\mathbb{R}^d)} \).

The local well-posedness result in theorem 0.1 was substantially improved to

**Theorem 0.4.** (0.1) is locally well-posed on \([-T,T]\) for \( u_0 \in L^2(\mathbb{R}^d) \), \( T(u_0) > 0 \), where \( T \) depends on the profile of \( u_0 \), not just its size.

*Proof:* See [6] and [7]. □

In this paper we sketch the proof of the natural extension of theorem 0.4,

**Theorem 0.5.** (0.1) is globally well-posed and scattering for all \( u_0 \in L^2(\mathbb{R}^d) \).\( \mu = +1 \). (0.1) is globally well-posed and scattering for \( \mu = -1 \), \( \|u_0\|_{L^2(\mathbb{R}^d)} < \|Q\|_{L^2(\mathbb{R}^d)} \) when \( u_0 \) is radial, \( d = 2 \).

*Proof:* See [24]. □

**Theorem 0.7.** (0.1) is globally well-posed and scattering for \( \mu = +1 \), \( u_0 \in L^2(\mathbb{R}^d) \) when \( u_0 \) is radial, \( d \geq 3 \).

*Proof:* See [37]. □

**Theorem 0.8.** (0.1) is globally well-posed and scattering for \( \mu = -1 \), \( \|u_0\|_{L^2(\mathbb{R}^d)} < \|Q\|_{L^2(\mathbb{R}^d)} \) when \( u_0 \) is radial, \( d \geq 3 \).

*Proof:* See [26]. □

**Conjecture:** If (0.1) is not globally well-posed and scattering, and \( \|u_0\|_{L^2(\mathbb{R}^d)} = \|Q\|_{L^2(\mathbb{R}^d)} \), then

\[
    u(t,x) = G \cdot e^{it}Q(x) \tag{0.13}
\]

where \( G = S^1 \times (0,\infty) \times \mathbb{R}^d \times \mathbb{R}^d \) is a group of symmetries acting on solutions to (0.1), or a conformal symmetry of (0.13).

\( G \) is generated by four symmetries which act on solutions of (0.1), multiplication,

\[
    u(t,x) \mapsto e^{i\theta}u(t,x), \tag{0.14}
\]

for \( \theta \in \mathbb{R} \), scaling,

\[
    u(t,x) \mapsto \frac{1}{\lambda^{d/2}}u(\frac{t}{\lambda^2}, \frac{x}{\lambda}), \tag{0.15}
\]

translation,

\[
    u(t,x) \mapsto u(t,x-x_0), \tag{0.16}
\]

and Galilean invariance

\[
    u(t,x) \mapsto e^{-it|\xi_0|^2}e^{ix\xi_0}u(t,x-2t\xi_0). \tag{0.17}
\]

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Let
\[
A_\mu(m) = \sup\{\|u\|_{L_1^2((0,\infty) \times \bR^d)} : \|u(t)\|_{L_2^2(\bR^d)} = m, \ u \text{ solves (0.1)}\}. \tag{0.18}
\]
To prove theorem 0.1 it suffices to prove \(A_\mu(m) < \infty\) for all \(m\) in the defocusing case and for \(m < \|Q\|_{L^2(\bR^d)}\) in the focusing case.

**Theorem 0.9.** \(A_\mu(m)\) is a continuous function of \(m\).

**Proof:** See [35].

This already implies a small data result for \(m < \epsilon(d, \mu)\) because \(A_\mu(0) = 0\). Moreover,

\[
\{m : A_\mu(m) = \infty\} \tag{0.19}
\]
is a closed set so if (0.19) is nonempty then it possesses a least element \(m_0\).

**Theorem 0.10.** Suppose

\[
\|u_n(t)\|_{L_2^2(\bR^d)} \nearrow m_0 \tag{0.20}
\]
\[
\|u_n\|_{L_1^2((t, \infty) \times \bR^d)} \nearrow \infty, \ \|u_n\|_{L_1^2((t, 0) \times \bR^d)} \nearrow \infty. \tag{0.21}
\]

Then \(u_n(t)\) has a subsequence that converges to \(u(t)\) in \(L^2(\bR^d)/G\), \(u(t) : I \subset \bR \to \bC\) is a solution to (0.1), \(I\) an open set.

\[
\|u(t)\|_{L_1^2((t, \infty) \times \bR^d)} = \|u(t)\|_{L_1^2((t, 0) \times \bR^d)} = \infty. \tag{0.22}
\]

Moreover, \(\{u(t) : t \in I\}\) lies in a compact subset of \(L^2(\bR^d)/G\). By the Arzela-Ascoli theorem there exist

\[
x(t), \xi(t) : I \to \bR^d, \tag{0.23}
\]
\[
N(t) : I \to (0, \infty), \tag{0.24}
\]
such that for all \(\eta > 0\) there exists \(C(\eta) < \infty\) such that

\[
\int_{|x-x(t)| \geq \frac{C(\eta)}{N(t)}} |u(t,x)|^2 \, dx + \int_{|\xi-\xi(t)| \geq C(\eta)N(t)} |\hat{u}(t,\xi)|^2 \, d\xi < \eta. \tag{0.25}
\]

**Proof:** See [36]. \(\square\)

Because \(u(t)\) lies in a precompact set we can take a limit of \(u(t_n), t_n \in I, \) in \(L^2(\bR^d)/G\) and obtain an even more special solution to (0.1).

**Theorem 0.11.** If theorem 0.1 fails then there exists a solution to (0.1) satisfying \(N(0) = 1, u(t)\) exists on \([0, \infty), \ N(t) \leq 1\) on \([0, \infty), \ x(0) = \xi(0) = 0, \)

\[
|\xi'(t)|, |N'(t)| \leq m_0 \ N(t)^3, \tag{0.26}
\]
\[
\|u\|_{L_1^2((0, \infty) \times \bR^d)} \leq \|u\|_{L_1^2((\inf(I), 0) \times \bR^d)} = \infty. \tag{0.27}
\]
Proof: See [25]. □

We consider two cases separately,

\[ \int_0^\infty N(t)^3 \, dt < \infty, \]  

and

\[ \int_0^\infty N(t)^3 \, dt = \infty. \]  

**Theorem 0.12.** If \( \int_0^\infty N(t)^3 \, dt = K \), then for all \( 0 \leq s < 1 + \frac{4}{d} \),

\[ \| u \|_{L_t^\infty H_x^s([0,\infty) \times \mathbb{R}^d)} \lesssim m_0, d K^s. \]  

This is enough to exclude (0.28) in the defocusing case and in the focusing case for mass below the mass of the ground state. In both cases \( E(u(0)) \geq \delta > 0 \) and (0.28) imply \( N(t) \to 0 \) as \( t \to \infty \). So when \( \xi(t) \equiv 0 \) it is easy to see \( N(t) \to 0 \), Sobolev embedding, and (0.30) imply

\[ E(u(t)) \to 0, \]  

which contradicts conservation of energy. In the general case when \( \xi(t) \) is free to move around, (0.26) implies that \( |\xi(t)| \lesssim m_0, d K \) for all \( t \in [0, \infty) \). For \( T \) sufficiently large \( N(T) \) is very small. After making a Galilean transformation sending \( \xi(T) \) to 0, this implies \( E(u(T)) \) is very small. Because \( |\xi(T)| \lesssim m_0, d K \) this transformation preserves (0.30). On the other hand, because \( L^p \) is Galilean invariant, after any Galilean transformation

\[ E(u(0)) \geq \delta > 0. \]  

This again contradicts conservation of energy.

Having completely ruled out case (0.28) we turn to case (0.29) when \( \mu = +1 \). We use the interaction Morawetz estimate

**Theorem 0.13.** If \( \mu = +1 \),

\[ \| |\nabla|^{\frac{3}{2}} d \frac{u(t, x)}{2} \| L^2_{t,x}([0,\infty) \times \mathbb{R}^d)} \lesssim m_0, d \int_0^T \partial_t M(t) \, dt, \]  

where

\[ M(t) = \int \frac{(x-y)^i}{|x-y|} \text{Im}[\bar{u}(t, x) \partial_j u(t, x)] |u(t, y)|^2 \, dx \, dy. \]  

Because the solution \( u(t) \) need not possess any additional regularity, we truncate in frequency.

**Theorem 0.14.** Suppose \( \int_0^T N(t)^3 \, dt = K \), choose \( C \) sufficiently large so that

\[ \int_0^T |\xi'(t)| \, dt \ll CK, \]  

which is always possible by (0.26). Let

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\[ M(t) = \int \frac{(x - y)^j}{|x - y|} Im[P_{\leq CK} \bar{u}(t, x) \partial_j P_{\leq CK} u(t, x)]|P_{\leq CK} u(t, y)|^2 dxdy. \] (0.36)

Then
\[ \int_0^T N(t)^3 dt \lesssim_{m_0, d} \int_0^T \partial_t M(t) dt, \] (0.37)
and since \( N(t) \leq 1 \) for \( t \in [0, \infty) \),
\[ |M(t)| \lesssim_{m_0, d} o(K). \] (0.38)
This rules out (0.29) in the case when \( \mu = +1 \) because \( K \) can be made arbitrarily large by taking \( T \) sufficiently large, giving the contradiction
\[ K \lesssim o(K). \] (0.39)

We now give a brief discussion of the proof of theorem 0.14. It is perhaps easiest to see that when \( d \geq 4 \), if \( u \) is a minimal mass blowup solution to (0.1),
\[ N(t)^3 \lesssim_{m_0, d} |||D|^{\frac{d-2}{2}}|u(t, x)|^2||_{L^2_x(R^d)}. \] (0.40)
Indeed, when \( d \geq 4 \),
\[ |||D|^{\frac{d-2}{2}}|u(t, x)|^2||_{L^2_x(R^d)} \sim_{m_0, d} \int \frac{1}{|x - y|^2} |u(t, x)|^2 |u(t, y)|^2 dxdy. \] (0.41)
The spatial concentration in (0.25) implies (0.40). Because most of the mass is contained in \( P_{\leq CK} \), we also have
\[ \int_0^T N(t)^3 dt \lesssim_{m_0, d} |||D|^{\frac{d-2}{2}}|P_{\leq CK} u(t, x)|^2||_{L^2_x([0, T] \times R^d)}. \] (0.42)
(0.38) follows from (0.25), \( N(t) \leq 1 \), and the fact that the interaction Morawetz estimates are Galilean invariant.

\[ i\partial_t (P_{\leq CK} u) + \Delta (P_{\leq CK} u) = \mu |P_{\leq CK} u|^{4/d}(P_{\leq CK} u) + \mu |P_{\leq CK} (|u|^{4/d} u)| - |P_{\leq CK} u|^{4/d}(P_{\leq CK} u). \] (0.43)
If we were able to drop
\[ \mu |P_{\leq CK} (|u|^{4/d} u) - |P_{\leq CK} u|^{4/d}(P_{\leq CK} u)| \] (0.44)
then the proof of (0.42) when \( \mu = +1 \) would be identical to the proof of (0.33). Therefore, most of the work in proving theorem 0.14 lies in showing that the error arising from (0.44) is bounded by \( o(K) \). In fact, the error estimates are quite robust.

**Theorem 0.15.** We can perform the same error estimates with \( \frac{(x - y)}{|x - y|} \) replaced with \( a(t, x - y) \), as long as
\[ |a(t, x)| \lesssim_{m_0, d} 1, \] (0.45)
when \( d = 2 \),
\[ |
\n\n\n\nIV–6
and when $d = 1$,

$$\|\nabla a(t, x)\|_{L^1_\infty(R^d)} \lesssim m_0, d_1,$$  \hspace{1cm} (0.47)

$$a(t, x) = -a(t, -x),$$  \hspace{1cm} (0.48)

and when $d = 2$,

$$\|\partial_t a(t, x)\|_{L^1(R^d)} \lesssim m_0, d_1.$$  \hspace{1cm} (0.49)

Therefore it remains to construct an interaction Morawetz potential bounded below by $N(t)^3$ and which satisfies (0.45) - (0.49). We do this only in the case when $d = 1$ and $u$ is radial. Suppose $\psi \in C^\infty(R)$, $\psi = \phi' \geq 0$, and

\begin{align*}
= x, & \quad |x| \leq 1, \\
\psi(x) = \frac{3}{2}, & \quad x > 2, \\
= -\frac{3}{2}, & \quad x < -2.
\end{align*}  \hspace{1cm} (0.50)

Then let

$$M(t) = R \int \psi(\frac{x\tilde{N}(t)}{R}) Im[\bar{u}\partial_x u](t, x) dx,$$  \hspace{1cm} (0.51)

such that for some $\delta > 0$

$$\delta N(t) \leq \tilde{N}(t) \leq N(t),$$  \hspace{1cm} (0.52)

and

$$\int_0^T |\tilde{N}'(t)| dt \leq \delta_1 \int_0^T \tilde{N}(t)N(t)^2 dt.$$  \hspace{1cm} (0.53)

Then

$$\partial_t M(t) = \tilde{N}(t) \int \phi(\frac{x\tilde{N}(t)}{R}) [\frac{1}{2} |\nabla u(t, x)|^2 - \frac{1}{6} |u(t, x)|^6] dx - \frac{R}{2} \int \phi''(\frac{x\tilde{N}(t)}{R}) |u(t, x)|^2 dx$$

$$+ C \frac{\tilde{N}(t)^3}{R^2} \int \phi''(\frac{x\tilde{N}(t)}{R}) |u(t, x)|^2 dx$$

$$+ R \int \phi(\frac{x\tilde{N}(t)}{R}) x\tilde{N}'(t) Im[\bar{u}\partial_x u](t, x) dx.$$  \hspace{1cm} (0.54)

We choose $\tilde{N}(t)$ to be a sufficiently slowly varying (0.53) envelope for $N(t)$ which allows us to absorb (0.55) into (0.54) for $R(m_0)$ sufficiently large and for $\delta_1(R)$ sufficiently small we can absorb (0.56) into (0.54). This completes the proof of the focusing case.

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References


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