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Boundary singularities of solutions to quasilinear elliptic equations

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Abstract

Asymptotic formulae for solutions to boundary value problems for linear and quasilinear elliptic equations and systems near a boundary point are discussed. The boundary is not necessarily smooth. The main ingredient of the proof is a spectral splitting and reduction of the original problem to a finite-dimensional dynamical system. The linear version of the corresponding abstract asymptotic theory is presented in our new book "Differential Equations with Operator Coefficients", Springer, 1999

In the article [W], S. Warschawski obtained an asymptotic formula for conformal mappings of curvilinear strips under rather weak restrictions to their boundaries. His proof is based on methods of geometric function theory. Here we state a corollary of Warschawski's result, which was the starting point of the present work.

Let Ω be the domain $\{z = x + iy : y > f(x)\}$ in the complex plane, where f is Lipschitz and $f(0) = 0$. By w we denote a conformal mapping of Ω onto the half-plane $\{w = u + iv : u > 0\}$, $w(0) = 0$. Suppose that $f'(x) \to 0$ as x approaches 0 except possibly for a set of measure zero and that

$$\int_0^1 (|f'(\rho)|^2 + |f'(-\rho)|^2)\frac{d\rho}{\rho} < \infty.$$

Then the real part of w admits the asymptotic representation as $z \to 0$:

$$u(z) \sim c\,(y - f(x)) \exp\Big\{ -\frac{1}{\pi}\int_{|z|}^1 (f(\rho) + f(-\rho))\frac{d\rho}{\rho^2}\Big\}, \tag{1}$$

where c is a constant. This relation can be interpreted as an asymptotic formula for a solution to the Laplace equation in a neighbourhood of the origin with zero Dirichlet data on the curve $\{z : y = f(x)\}$.

In the present lecture we state asymptotic representations similar to (1) for solutions to the Dirichlet problem for arbitrary order linear and even quasilinear elliptic equations and systems in the multi-dimensional Lipschitz domain

$$\Omega = \{x = (x', x_n) \in \mathbb{R}^n : x_n > f(x')\}\ .$$

Let $f(0) = 0$ and

$$\int_0^1 ||\nabla f||^2_{L^\infty(B_\rho^{n-1})} \frac{d\rho}{\rho} < \infty , \tag{2}$$

with B_ρ^k denoting the k–dimensional ball with radius ρ centred at the origin.

The representations just mentioned are examples of application of a recently developed asymptotic theory of nonlinear abstract higher order ordinary differential equations. ¿From 1990 we worked on an asymptotic theory of arbitrary order *linear* differential equations with unbounded operator coefficients in a Hilbert or Banach space. The impetus was given by our previous and simultaneous studies of singularities of solutions to elliptic boundary value problems (see [KMR]). The first real publication summarizing our linear abstract theory has just appeared as the Springer book [KM1] (previously we wrote seven preprints concerning this topic). In Introduction to the book we mentioned that reach possibilities for generalizations to *nonlinear* operator equations were left aside. Some of these possibilities are discussed in what follows.

We study solutions to the equation

$$\mathcal{A}(D_t)u(t) = \sum_{j=0}^{q} D_t^{q-j} \mathcal{N}_j(t; u(t), \dots , D_t^{\ell---q}u(t)) \tag{3}$$

on a semiaxis $t > t_0$, where $\mathcal{A}(D_t)$ is an ordinary differential operator of order ℓ with unbounded operator coefficients in a Hilbert space and $\mathcal{N}_0, \dots ,\mathcal{N}_q$ are nonlinear operators. We write the right-hand side in this form in order to cover more general nonlinearities: the operators $\mathcal{N}_j$ are not assumed to be differentiable. Our conditions on $\mathcal{A}(D_t)$ are dictated by an analogy with linear elliptic operators written in the variational form and are the same as in [KM1].

Our main concern is with the asymptotic behaviour of solutions as $t \to +\infty$. We show that for a certain class of equations (3) the question of asymptotics can be reduced to that for a finite dimensional dynamical system perturbed by a "weak" non-local nonlinear operator. This is a far-reaching generalization of a similar result for linear differential equations with operator coefficients obtained in [KM1] (see Ch. 13). However, since the right-hand side is now written in the generic divergence form, our present result is new also for the linear case and, apparently, even for linear ordinary differential equations with scalar coefficients.

In order to avoid technicalities we give a very approximate description of a general result concerning a solution $u(t)$ to (3). Let u be subject to a growth condition of the type

$$||u||_{W^{\ell-q}(t,t+1)} \leq M(t) \quad \text{for large positive } t, \tag{4}$$

where $W^{\ell-q}$ is a Sobolev space and the majorant $M(t)$ behaves like $\exp(-k_0 t)$ in a certain rough sense. Estimates of this type are usually obtained in applications by using specific features of the equation: one relies upon monotonicity properties of differential operators, the maximum modulus principle and more refined tricks. Dealing with an abstract theory we take (4) for granted.

Under some natural assumptions on the nonlinearity we obtain the representation

$$\operatorname{col}(u(t),\dots,D_t^{\ell-q-1}u(t))=\sum_{s=1}^{\kappa}h_s(t)\operatorname{col}(U_s(t),\dots,D_t^{\ell-q-1}U_s(t))+\mathbf{v}(t). \quad (5)$$

The vector functions U_s are solutions to $\mathcal{A}(D_t)U(t)=0$ of the form $\exp(i\lambda_\nu t)P(t)$ with λ_ν being eigenvalues of the pencil $\mathcal{A}(\lambda)$ on the line $\Im\lambda=k_0$ and with P being polynomials with constant vector coefficients. The vector $\vec{h}=(h_1,\dots,h_\kappa)$ satisfies the finite dimensional perturbed dynamical system

$$D_t\vec{h}(t)=\mathbf{N}(t;\vec{h}(t))+\mathbf{K}[\vec{h}](t)\ . \quad (6)$$

For the components of $\mathbf{N}$ we have the representation

$$\mathbf{N}_k[\vec{h}](t)=\sum_{j=0}^{q}\langle\mathcal{N}_j(t;\sum_{s=1}^{\kappa}h_s(t)U_s(t),\dots,\sum_{s=1}^{\kappa}h_s(t)D_t^{\ell-q}U_s(t))|D_t^{q-j}V_k\rangle$$

where V_k are solutions of the adjoint equation $\mathcal{A}^*(D_t)V(t)=0$ which are connected with U_s by a biortogonality condition. Here, $\langle\cdot|\cdot\rangle$ is the scalar product. By $\mathbf{K}$ we denote a non-local nonlinear operator.

The vector function $\mathbf{v}$ can be regarded as a remainder term. We give estimates which show that $\mathbf{K}$ and $\mathbf{v}$ are weak in a certain sense.

Our main technical tool is a comparison principle which shows that solutions to a certain ordinary differential equation majorize solutions of equation (3) (compare with [KM1], Sect.5, and [KM2]). It is this principle that we use to obtain estimates for the vector function $\mathbf{v}$ and the operator $\mathbf{K}$.

System (6) is the corner stone of our asymptotic theory. On one hand, it can be applied to construct solutions of (3) with the vector $\vec{h}(t)$ asymptotically close to a solution of the dynamical system

$$D_t\vec{\chi}(t)=\mathbf{N}(t;\vec{\chi}(t))\ . \quad (7)$$

On the other hand, one can try to show that solutions to (3) subject to the growth restriction (4) have the asymptotic representation (5), where the vector $\vec{h}$ satisfies (7).

Consider the differential equation in the domain $B_1^n\cap\mathbb{R}_+^n$:

$$\mathcal{L}_{2m}(\partial_x)u(x)=\sum_{|\alpha|\le m}(-\partial_x)^\alpha\big(N_\alpha(x,\{\,\partial_x^\gamma u(x)\,\}_{|\gamma|\le m})\big), \quad (8)$$

where $\partial_x=\nabla$. By $\mathcal{L}_{2m}(\xi)$ we mean a homogeneous polynomial of degree $2m$ with real constant coefficients.

We assume that $(-1)^m\mathcal{L}_{2m}(\xi)$ is positive for $\xi\in\mathbb{R}^n\setminus\mathcal{O}$. Without loss of generality we suppose that the coefficient of $\mathcal{L}_{2m}(\partial_x)$ in $\partial_{x_n}^{2m}$ is equal to $(-1)^m$. By $N_\alpha=N_\alpha(x,\{Y_\beta\}_{|\beta|\le m})$ we denote functions subject to the Carathéodory conditions, that is these functions are measurable in x for all $\{Y_\beta\}_{|\beta|\le m}$ and continuous in $\{Y_\beta\}_{|\beta|\le m}$ for almost all x.

The solution u, which is understood in a variational sense, belongs to the local Sobolev space $W^{p,m}_{\text{loc}}(\overline{B^n_1 \cap \mathbb{R}^n_+} \setminus \mathcal{O})$, $p \in (0,\infty)$, and is subject to zero Dirichlet conditions on $(B^n_1 \cap \partial\mathbb{R}^n_+) \setminus \mathcal{O}$.

Under some restrictions on the operators and solution we obtain the asymptotic formula

$$u(x) \sim X(r)x^m_n \ ,$$

where the function X satisfies the ordinary differential equation

$$\partial_r X(r) + r^{n-1} \int_{S^{n-1}_+} \sum_\alpha N_\alpha(x,\, X(r)\{\, \partial^\gamma_x x^m_n \,\}_{|\gamma|\le m}) \partial^\alpha_x E(x) d\theta = 0 \ , \qquad (9)$$

where S^{n-1}_+ is the upper unit hemisphere and $x = r\theta$, $r = |x|$. By E we denote the solution of

$$\mathcal{L}_{2m}(\partial_x) u(x) = 0 \qquad (10)$$

in $\mathbb{R}^n_+$, positive homogeneous of degree $m-n$ and subject to the Dirichlet conditions on the hyperplane $x_n = 0$:

$$\partial^j_{x_n} E = 0 \quad \text{for} \quad 0 \le j \le m-2 \ , \quad \text{and} \quad \partial^{m-1}_{x_n} E = \delta(x') \ ,$$

where δ is the Dirac δ-function. Up to the change of variables $t = \log r^{-1}$ this equation is a special case of the dynamical system (7).

Let us discuss the special case when equation (8) is linear, that is we consider the Dirichlet problem

$$\mathcal{L}_{2m}(\partial_x) u = \sum_{|\alpha|,|\beta|\le m} (-\partial_x)^\alpha \big(N_{\alpha\beta}(x) \partial^\beta_x u\big) \quad \text{in } \mathbb{R}^n_+ \cap B_1, \qquad (11)$$

$$\partial^k_{x_n} u(x',0) = 0 \quad \text{for } k = 0,1,\dots,m-1 \text{ and } x' \in (\mathbb{R}^{n-1} \cap B_1) \setminus \mathcal{O}. \qquad (12)$$

Let $N_{\alpha\beta}$ satisfy

$$\int_0^1 \sum_{|\alpha|,|\beta|\le m} ||x_n^{2m-|\alpha+\beta|} N_{\alpha\beta}||^2_{L^\infty(\Omega_\rho)} \rho^{-1} d\rho < \infty \ , \qquad (13)$$

where $\Omega_\rho = \{x \in \Omega : \rho > |x| > \rho/e\}$. Now equation (9) is written as

$$\partial_r X(r) + r^{n-1} \int_{S^{n-1}_+} \sum_{|\alpha|,|\beta|\le m} N_{\alpha\beta}(x) \partial^\beta_x x^m_n \, \partial^\alpha_x E(x) d\theta \, X(r) = 0$$

and therefore any solution of (11), (12) subject to

$$|u(x)| \le C|x|^{m-n-1+\varepsilon} \quad \text{with } \varepsilon > 0 \qquad (14)$$

admits one of the following asymptotic representations:

Either

$$u(x) \sim c\,E(x)\exp\Big\{\int_{|x|<|\xi|<1}\sum_{|\alpha|,|\beta|\le m} N_{\alpha\beta}(\xi)\partial_\xi^\beta E(\xi)\ \partial_\xi^\alpha \xi_n^m d\xi\Big\}$$

or

$$u(x) \sim c\,x_n^m \exp\Big\{-\int_{|x|<|\xi|<1}\sum_{|\alpha|,|\beta|\le m} N_{\alpha\beta}(\xi)\partial_\xi^\beta \xi_n^m\ \partial_\xi^\alpha E(\xi)d\xi\Big\}\,.$$

We formulate two special cases which seem to be of independent interest.

Consider the equation

$$--\sum_{i,j=1}^{n}\partial_{x_i}(a_{ij}(x)\partial_{x_j}u)=0 \quad \text{in } \mathbb{R}_+^n\cap B_1$$

complemented by the boundary condition

$$u(x',0)=0 \quad \text{for } x'\in\mathbb{R}^{n-1}\cap B_1.$$

Let

$$\int_0^1\sum_{i,j=1}^{n}||a_{ij}(\cdot)-\delta_i^j||^2_{L^\infty(\Omega_\rho)}\rho^{-1}d\rho<\infty\,.$$

Then any solution u subject to

$$|u(x)|\le C\rho^{1-n+\varepsilon} \tag{15}$$

admits the asymptotic representation

$$u(x)\sim c\,x_n\exp\Big\{\frac{\Gamma(n/2)}{\pi^{n/2}}\int_{|x|<|\xi|<1}\sum_{i=1}^{n}(a_{in}(\xi)-\delta_i^n)\partial_{\xi_i}(\xi_n|\xi|^{-n})d\xi\Big\}\,.$$

Another example is provided by the Schrödinger equation

$$(i\nabla+\vec{m}(x))^2u-P(x)u=0 \quad \text{in } \mathbb{R}_+^n\cap B_1$$

complemented by zero Dirichlet condition on the hyperplane $x_n=0$. Here $\vec{m}$ is a magnetic vector potential and P is an electric potential. The condition (13) becomes

$$\int_0^1\sup_{\xi\in\Omega_\rho}(\xi_n|\vec{m}(\xi)|+\xi_n^2|P(\xi)|)^2\frac{d\rho}{\rho}<\infty\,.$$

Then all solutions subject to (15) admit the asymptotic representation

$$u(x)\sim c\,x_n\exp\Big\{-\frac{\Gamma(n/2)}{\pi^{n/2}}\int_{|x|<|\xi|<1}\Big(P(\xi)-|\vec{m}|^2--\frac{n(\vec{m}(\xi),\xi)}{|\xi|^2}i\Big)\frac{\xi_n^2}{|\xi|^n}d\xi\Big\}\,.$$

Now we come back to the Lipschitz domain Ω and consider equation (8) in $B_1^n\cap\Omega$. Let u be a variational solution of (8) in the local Sobolev space $W_{\text{loc}}^{p,m}(\overline{B_1^n\cap\Omega\setminus\mathcal{O}})$ and subject to zero Dirichlet conditions on $(B_1^n\cap\partial\Omega)\setminus\mathcal{O}$.

The results obtained are new even for solutions to the linear equation (10). Therefore we begin by describing the asymptotic behaviour of solutions to (10). We show that any solution to equation (10) in $B_1^n \cap \Omega$ satisfying the zero Dirichlet conditions on $(B_1^n \cap \partial\Omega) \setminus \mathcal{O}$ and the estimate (14) is subject to the following asymptotic alternatives:

Either

$$u(x) \sim c\, E(x', x_n - f(x')) \exp\left\{ \int_{|x|<|\xi'|<1} f(\xi') \partial_{\xi_n}^m E(\xi', 0) d\xi' \right\}$$

or

$$u(x) \sim c\, (x_n - f(x'))^m \exp\left\{ - \int_{|x|<|\xi'|<1} f(\xi') \partial_{\xi_n}^m E(\xi', 0) d\xi' \right\}, \tag{16}$$

where $c = \text{const}$.

In the special case of the polyharmonic operator $\mathcal{L}_{2m} = (-\Delta)^m$ the above alternatives become:

Either

$$u(x) \sim c\, \frac{(x_n - f(x'))^m}{|x|^n} \exp\left\{ m\, \frac{\Gamma(n/2)}{\pi^{n/2}} \int_{|x|<|\xi'|<1} f(\xi') \frac{d\xi'}{|\xi'|^n} \right\}$$

or

$$u(x) \sim c\, (x_n - f(x'))^m \exp\left\{ -m\, \frac{\Gamma(n/2)}{\pi^{n/2}} \int_{|x|<|\xi'|<1} f(\xi') \frac{d\xi'}{|\xi'|^n} \right\} .$$

If $m = 1$ and $n = 2$ the last formula coincides with the Warschawski asymptotic formula (1).

We give two examples of application of our asymptotic theory to quasilinear partial differential equations. Consider the equation

$$\mathcal{L}_{2m}(\partial_x) u = |\nabla_k u(x)|^{(m+n)/(k+n-m)} \quad \text{in } B_1^n \cap \Omega.$$

where k is a non-negative integer such that $m - n < k \le m$. Now, we deal with an arbitrary solution $u \in W_{\text{loc}}^{p,m}(\overline{B_1^n \cap \Omega} \setminus \mathcal{O})$ satisfying zero Dirichlet conditions on $(B_1^n \cap \partial\Omega) \setminus \mathcal{O}$. We replace (2) by the more stringent condition

$$\big|\log|x'|\big|\, |\nabla f(x')| \le C \quad \text{for small } |x|. \tag{17}$$

Then one of the following alternatives is valid:

Either u satisfies (16) or

$$u(x) \sim h(|x|) E(x', x_n - f(x')) ,$$

where h is a solution to the equation

$$h'(r) + \frac{\overline{f}(r)}{r^2} h(r) - \frac{a}{r} |h(r)|^{(m+n)/(k+n-m)} = 0$$

with

$$\overline{f}(r) = \int_{S^{n-2}} f(r\theta')(\partial_{x_n}^m E)(\theta', 0) d\theta' .$$

and

$$a = \int_{S_+^{n-1}} |(\nabla_k E)(\theta)|^{(m+n)/k+n-m} \theta_n^m d\theta \, .$$

The second example deals with an arbitrary solution $\vec{u}$ to the equation

$$\Delta \vec{u} = |x|^{-4} |\vec{u}|^2 \vec{u}$$

in $B_1^n \cap \Omega$, which is continuous in $\overline{B_1^n \cap \Omega} \setminus \mathcal{O}$ and vanishes on $(B_1^n \cap \partial\Omega) \setminus \mathcal{O}$. Using our asymptotic scheme we show that under condition (17)

$$\vec{u}(x) \sim \vec{e}\, h(|x|)(x_n - f(x'))$$

where $\vec{e}$ is a constant unit vector and h is a solution of the equation

$$h'(r) - \frac{\overline{f}(r)}{r^2} h(r) - \frac{a}{r} h^3(r) = 0 \, .$$

Here

$$\overline{f} = \frac{\Gamma(n/2)}{\pi^{n/2}} \int_{S^{n-2}} f(r\theta') d\theta'$$

and

$$a = \int_{S_+^{n-1}} \theta_n^4 d\theta \, .$$

In conclusion we add that the same ideas led us in [KM3] to an asymptotic description of all solutions to the Neumann problem for the two-dimensional Riccati equation:

$$\Delta u + \alpha(x)(\partial_{x_1} u)^2 + 2\beta(x)\partial_{x_1} u\, \partial_{x_2} u + \gamma(x)(\partial_{x_2} u)^2 = 0 \tag{18}$$

in the sector $K_\delta = \{x = (x_1, x_2) \in \mathbb{R}^2 : 0 < r < \delta, \theta \in (0, \varphi)\}$, where (r, θ) are the polar coordinates of x, and $\varphi \in (0, 2\pi]$,

$$\partial_\theta u|_{\theta=0} = \partial_\theta u|_{\theta=\varphi} = 0 \quad \text{for } r < \delta.$$

Here α, β and γ are measurable functions. We suppose that for almost all $x \in K_\delta$ and for all $(\xi_1, \xi_2) \in \mathbb{R}^2$

$$\lambda\left(\xi_1^2 + \xi_2^2\right) \le \alpha(x)\xi_1^2 + 2\beta(x)\xi_1\xi_2 + \gamma(x)\xi_2^2 \le \Lambda\left(\xi_1^2 + \xi_2^2\right) \tag{19}$$

with positive constants λ and Λ. We assume that u belongs to the Sobolev space $W^{2,2}(\overline{K}_\delta \setminus \mathcal{O})$.

There exist two possibilities: either u is unbounded and then

$$u(x) = Q(r) + c_* + o(1), \tag{20}$$

where

$$Q(r) = \varphi \int_r^{\delta/e} \frac{ds}{s} \Big(\int\int_{x \in K_\delta \setminus K_s} \frac{\alpha(x)x_1^2 + 2\beta(x)x_1x_2 + \gamma(x)x_2^2}{|x|^4} dx_1 dx_2 \Big)^{-1},$$

or u is bounded and has the same asymptotics

$$u(x) = c_0 + c_1 r^{\pi/\varphi} \cos(\pi\theta/\varphi) + o(r^{\pi/\varphi}) \tag{21}$$

as in the case of the Neumann problem for $\Delta u = 0$. Here c_*, c_0 and c_1 are real constants.

Clearly,

$$\lambda \log\log \frac{\delta}{r} \le Q(r) \le \Lambda \log\log \frac{\delta}{r} \quad \text{for } r \le \delta/e. \tag{22}$$

If the coefficients α, β and γ are constant we write the asymptotic expansion for unbounded solutions:

$$u(x) \sim d \log\log r^{-1} + c_* + \sum_{k=1}^{\infty} \frac{P_k(\log\log r^{-1}, \theta)}{(\log r^{-1})^k},$$

where

$$d = \Big(\frac{\alpha+\gamma}{2} + \beta\,\frac{\sin^2\varphi}{\varphi} + \frac{\alpha-\gamma}{4}\,\frac{\sin 2\varphi}{\varphi}\Big)^{-1},$$

and $P_k(\tau, \theta)$ are polynomials of degree $\le k$ in τ whose coefficients are smooth functions of $\theta \in [0, \varphi]$. If u is bounded it admits the asymptotic representation

$$u(x) \sim c_0 + \sum_{k=1}^{\infty} r^{k\pi/\varphi} p_{k-1}(\log r, \theta), \tag{23}$$

where $c_0 = \text{const}$ and p_k are polynomials in the first argument with smooth coefficients on $[0, \varphi]$.

These results and their proofs can be extended to the case when $\mathcal{O}$ is the center of the disk $K_\delta = \{x : r < \delta\}$. One should only put $\varphi = \pi$ in (21) and (23). In other words we also describe the asymptotic behaviour of solutions to equation (18) which are either bounded at $\mathcal{O}$ or have an isolated singularity there.

It is worth noting that equation (18) and the Neumann conditions as well as assumption (19) about the coefficients α, β and γ are preserved under conformalmappings. Therefore (20) and (21) along with asymptotics of conformal mappings (see [W]) imply asymptotic representations of solutions at infinity and near boundary singularities other than corners, for example, cusps.

The requirement (19) can not be removed. In fact, the function $\mathcal{U}(x) = \log r^{-1} + \cos\theta$ is a solution of the homogeneous Neumann problem for the equation

$$\Delta\mathcal{U} + \frac{\cos\theta}{1+\sin^2\theta}|\nabla\mathcal{U}|^2 = 0$$

in the upper half-plane. By (22) the singularity of this solution is stronger than that in (21).

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