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An Hadamard maximum principle for the bilaplacian on hyperbolic manifolds

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Abstract

We prove the existence of a maximum principle for operators of the type $\Delta\omega{-}1\Delta$, for weights ω with $\log\omega$ subharmonic. It is associated with certain simply connected subdomains that are produced by a Hele-Shaw flow emanating from a given point in the domain. For constant weight, these are the circular disks in the domain. The principle is equivalent to the following statement. THEOREM. Suppose ω is logarithmically subharmonic on the unit disk, and that the weight times area measure is a reproducing measure (for the harmonic functions). Then the Green function for the Dirichlet problem associated with $\Delta\omega^{-1}\Delta$ on the unit disk is positive.

1. The classical maximum principle.

Let Δ be the Laplace operator in $\mathbb{R}^n$ $(n = 1, 2, 3, \dots)$. We say that a real-valued function u is harmonic on an open subset of $\mathbb{R}^n$ if $\Delta u = 0$ there, and subharmonic if $\Delta u \geq 0$. A domain in $\mathbb{R}^n$ is an open and connected set. The classical maximum principle for subharmonic functions can be given the following formulation. Let D be a bounded domain in $\mathbb{R}^n$, and u a function continuous on the closure of D. We then have the implication

$$0 \leq \Delta u|_D \text{ and } u|_{\partial D} \leq 0 \implies u|_D \leq 0. \qquad \text{(MP:}\Delta\text{)}$$

Moreover, unless $u|_D = 0$, the conclusion can be sharpened to $u|_D < 0$.

2. A biharmonic maximum principle.

It is natural to try to extend the classical maximum principle to higher order elliptic operators: we focus on the simplest example, the bilaplacian Δ^2. In the same way that physically, the laplacian corresponds to a membrane, the bilaplacian corresponds to a plate (there is also a connection with creeping flow). In view of the nature of the boundary data for the Dirichlet problem, the maximum principle we are looking for necessarily will involve two inequalities along the boundary of the subdomain D, one for the functions, and another for the normal derivatives. We first need some notation. A real-valued function u on a domain Ω is *biharmonic* provided that $\Delta^2 u = 0$ there, and *sub-biharmonic* if $\Delta^2 u \leq 0$ (one should think of Δ as a negative operator, which is the reason why the inequality is switched as compared with the definition of subharmonic functions). In the following we shall restrict our attention to the case of the plane $\mathbb{R}^2$, which is identified with $\mathbb{C}$, the complex plane. Around 1900, it was known – more or less – that a variant of a maximum principle can be formulated for circular disks. Let D be a circular disk and u a C^1-smooth function on the closure of D. The maximum principle reads

$$\Delta^2 u|_D \leq 0, \quad u|_{\partial D} \leq 0, \quad \text{and} \quad \frac{\partial u}{\partial n}\Big|_{\partial D} \leq 0 \implies u|_D \leq 0, \qquad \text{(MP:}\Delta^2\text{)}$$

whereby the normal derivative is calculated in the interior direction. Actually, unless $u|_D = 0$, we have $u|_D < 0$. Let Γ_D denote the Green function for the Dirichlet problem associated with Δ^2 on D: for fixed $\zeta \in D$, the function $\Gamma_D(\cdot, \zeta)$ vanishes along with its normal derivative on ∂D, and $\Delta^2\Gamma_D(\cdot, \zeta)$ equals the unit point mass at ζ. The above maximum principle (MP:Δ^2) then expresses the following three basic facts:

$$0 < \Gamma_D(z, \zeta), \qquad (z, \zeta) \in D \times D, \tag{2.1}$$

$$0 < \Delta_z \Gamma_D(z, \zeta), \qquad (z, \zeta) \in \partial D \times D, \tag{2.2}$$

and

$$\frac{\partial}{\partial n(z)} \Delta_z \Gamma_D(z, \zeta) < 0, \qquad (z, \zeta) \in \partial D \times D. \tag{2.3}$$

These properties are easily verified by computation. In fact, with the normalizations used here, the Green function $\Gamma = \Gamma_{\mathbb{D}}$ for the unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ is expressed by

$$\Gamma(z, \zeta) = |z - \zeta|^2 \log \left| \frac{z - \zeta}{1 - z\bar{\zeta}} \right|^2 + (1 - |z|^2)(1 - |\zeta|^2), \qquad (z, \zeta) \in \mathbb{D} \times \mathbb{D}. \tag{2.4}$$

It should be mentioned that a local analysis of the behavior near the boundary shows that (2.2) is an immediate consequence of (2.1), at least if we replace the sign "$<$" with "$\leq$". The connection between (MP:Δ^2) and (2.1)–(2.3) is apparent from the symmetry of Γ_D together with Green's formula:

$$u(z) = \int_D \Gamma_D(z, \zeta)\, \Delta^2 u(\zeta)\, d\Sigma(\zeta) + \frac{1}{2} \int_{\partial D} \Big(\Delta_\zeta \Gamma_D(z, \zeta)\, \frac{\partial u}{\partial n}(\zeta) - \frac{\partial}{\partial n(\zeta)} \Delta_\zeta \Gamma_D(z, \zeta)\, u(\zeta) \Big)\, d\sigma(\zeta), \qquad z \in D, \tag{2.5}$$

where $d\Sigma$ is area measure, normalized by the factor π^{-1}, and $d\sigma$ is one-dimensional Lebesgue measure, normalized by the factor $(2\pi)^{-1}$.

Notation: We have normalized the laplacian Δ acting over the plane: it is the operator

$$\Delta_z = \frac{1}{4} \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right), \qquad z = x + iy \in \mathbb{C}.$$

A locally summable functions f on some domain Ω in the plane is equated with a distribution by the dual action

$$\langle \varphi, f \rangle = \int_\Omega \varphi(z)\, f(z)\, d\Sigma(z),$$

for test functions φ: compactly supported C^∞-smooth functions on Ω.

3. Hyperbolic geometry.

In his treatise on *plaques élastiques encastrées* ([3], pp. 515–641), Jacques Hadamard suggests the possibility of a maximum principle of the type (MP:Δ^2) for more general subdomains D – in fact, he conjectured that it would hold for all convex D. That was later disproved in a series of papers of Duffin, Lœwner, and Garabedian (see, for instance, [2]). With hindsight, it is possibly to intuitively feel why the maximum principle (MP:Δ^2) fails in such generality; the circular disks are natural domains for the bilaplacian. Let us keep the circles – to be more precise, we single out the Gauss mean value property which they possess – but consider instead much more general geometries.

Let Ω be a simply connected two-dimensional Riemannian manifold with metric ds. Then topologically, Ω is either the sphere, the plane, or the open unit disk, by the Kœbe theorem. We shall be interested in hyperbolic metrics, in which case we have only the plane or the disk as alternatives. We choose the disk, because it can be used to give local coordinates of the sphere and the plane. Then Ω equals the unit disk $\mathbb{D}$, and we have a global coordinate chart. The metric ds can be expressed as

$$ds(z)^2 = \alpha(z)\,dx^2 + \beta(z)\,dy^2 + \gamma(z)\,dxdy, \qquad z = x + iy \in \mathbb{D},$$

where α, β, and γ are assumed smooth, and the associated 2×2 matrix is strictly positively definite. The basic "circular" neighborhoods of points need not look circular in this first choice of coordinates (they are instead elliptic), but this is easily remedied. We find a quasi-conformal mapping from $\mathbb{D}$ to itself which straightens out the ellipses to circles, and in the new coordinates, the metric takes the form

$$ds(z)^2 = \varrho(z)^2\,|dz|^2 = \varrho(z)^2\,(dx^2 + dy^2), \qquad z = x + iy \in \mathbb{D},$$

for some smooth strictly positive function ϱ on $\mathbb{D}$. The Laplace-Beltrami operator Δ on the Riemannian manifold is very simple to express in these coordinates:

$$\Delta = \frac{1}{\varrho(z)^2}\,\Delta.$$

The distributional derivative $\Delta \log \varrho$ turns out to be independent of conformal coordinate changes, and, in fact, $\mu = -\Delta \log \varrho$ represents a fundamental geometric quantity: *the local distribution of Gaussian curvature*. Let us assume that the Riemannian manifold is *hyperbolic*, meaning that the Gaussian curvature is negative everywhere. Then μ is a negative locally bounded Borel measure on $\mathbb{D}$, and the μ-measure of a subset tells us how curved the space is on the set. It is natural to consider first smooth metrics, and then consider more rugged metrics as a limit of smooth ones. The natural generalization of the planar maximum principle is

$$\Delta^2 u|_D \le 0, \quad u|_{\partial D} \le 0, \quad \text{and} \quad \frac{\partial u}{\partial n}\Big|_{\partial D} \le 0 \implies u|_D \le 0, \qquad (\text{MP}{:}\Delta^2)$$

where D is a smooth precompact subdomain of $\mathbb{D}$, and the function u is assumed smooth on the closure of D. Of course we cannot expect the above to hold for all subdomains D, but rather, we should look for the appropriate generalization of the circular disks. The first attempt probably would be to consider the metric disks, but the above maximum principle definitely fails for them (a counterexample can be worked out on the basis of Miroslav Engliš's work [1]). Instead we focus on the mean value property. Bernard Epstein [6] has shown that a domain D in the complex plane with $z_0 \in D$ and possessing the mean value property for some fixed r, $0 < r < +\infty$,

$$\int_D h(z)\,d\Sigma(z) = r^2 h(z_0),$$

with h ranging over all bounded harmonic functions on D, necessarily is a circular disk centered at z_0 with radius r. A precompact subdomain D of the hyperbolic manifold $\mathbb{D}$ is said to be a mean value disk – "centered" at $z_0 \in D$ with "radius" r, $0 < r < +\infty$ – provided that

$$\int_D h(z)\,\varrho(z)^2\,d\Sigma(z) = r^2 h(z_0),$$

holds for all bounded harmonic functions h on D. The mean value disks turn out to be uniquely determined by the parameters z_0 and r, just like the circles, and they are simply connected (this need not be true without the assuption of hyperbolicity). They are the result of a physical process, a *Hele-Shaw flow* on the hyperbolic manifold. The Hele-Shaw flow models how the free boundary evolves between an incompressible viscous Newtonian fluid and vacuum, which occupy the space between two parallel, narrowly separated infinitely extended surfaces (parallel to the manifold), as fluid is injected at a constant rate at the source point z_0. The time parameter corresponds to r^2, and initially, the space between the surfaces is empty.

The maximum principle (MP:$\Delta\omega^{-1}\Delta$) probably holds true holds for mean value disks D. The proof of this statement, however, remains to be found. Nevertheless, *for mean value disks D, we have found the weaker principle* [5]

$$\Delta^2 u|_D \le 0, \quad u|_{\partial D} = 0, \quad \text{and} \quad \left.\frac{\partial u}{\partial n}\right|_{\partial D} \le 0 \implies u|_D \le 0, \qquad (\mathrm{MP}':\Delta^2)$$

assuming u is smooth on D. It is convenient to state the result for D equal to $\mathbb{D}$, the unit disk. Then the metric $\varrho(z)\,|dz|$ should have the reproducing property

$$\int_{\mathbb{D}} h(z)\,\varrho(z)^2\,d\Sigma(z) = h(0) \tag{3.1}$$

for all bounded harmonic functions h on $\mathbb{D}$. The Green function for the Dirichlet problem associated with the weighted biharmonic operator $\Delta\,\varrho^{-2}\Delta$ on the unit disk $\mathbb{D}$ is denoted by Γ_ϱ. The main result – equivalent to (MP$'$:Δ^2) – is the following.

THEOREM 3.1 *Suppose the metric $\varrho(z)\,|dz|$ is hyperbolic on $\mathbb{D}$, and has the above reproducing property (3.1). Then $0 \le \Gamma_\varrho|_{\mathbb{D}\times\mathbb{D}}$.*

The statement of the theorem is false if we keep the reproducing property but scrap the logarithmic subharmonicity. These examples also show that hyperbolic geometry lies on the "right" side of flat geometry, in the sense that for elliptic Riemannian manifolds (with *positive* Gaussian curvature) there is no general theorem of the above type.

4. Ideas of the proof.

It is important to note that since $\Delta\varrho^{-2}\Delta\Gamma_\varrho(\cdot,\zeta)$ is a unit point mass at ζ, it follows that

$$\Delta_z\Gamma_\varrho(z,\zeta) = \varrho(z)^2\left(G(z,\zeta) + H_\varrho(z,\zeta)\right), \qquad (z,\zeta)\in\mathbb{D}\times\mathbb{D},$$

where

$$G(z,\zeta) = \log\left|\frac{z-\zeta}{1-z\bar\zeta}\right|^2$$

is the Green function for the laplacian, and the function $H_\varrho(z,\zeta)$ is harmonic in z. We shall call the kernel function H_ϱ the *harmonic compensator*, because it solves the balayage problem

$$\int_{\mathbb{D}} \left(G(z,\zeta) + H_\varrho(z,\zeta)\right) h(z)\,\varrho(z)^2\,d\Sigma(z) = 0, \qquad \zeta\in\mathbb{D},$$

for all bounded harmonic functions h on $\mathbb{D}$. An approximation argument allows us to assume substantial regularity of the metric function ϱ, such as real analyticity up to the boundary. If the Green function Γ_ϱ is positive, then a local analysis near the boundary shows that $\Delta_z\Gamma_\varrho(z,\zeta)$ is positive on $\mathbb{T}\times\mathbb{D}$, and using the harmonicity of H_ϱ in the first variable, it follows that H_ϱ is positive throughout $\mathbb{D}\times\mathbb{D}$. It is much less obvious that if the harmonic compensator is positive for a certain family of weight functions of the same type as ϱ, then we can go the other way around and obtain the positivity of Γ_ϱ. This is done with the help of a variational technique due to Hadamard (see [3, 4]), along domains $D(r)$ given by Hele-Shaw flow on the hyperbolic manifold, with injection point $z_0 = 0$ and time parameter r^2, $0 < r \le 1$, ending with $D(1) = \mathbb{D}$.

The harmonic compensator is related to the reproducing kernel function Q_ϱ for the space $HP^2(\mathbb{D},\varrho)$ obtained as the closure of the harmonic polynomials with respect to the norm of $L^2(\mathbb{D},\varrho)$,

$$\|f\|_\varrho = \left(\int_{\mathbb{D}} |f(z)|^2\varrho(z)^2\,d\Sigma(z)\right)^{\frac{1}{2}}.$$

More precisely,

$$\Delta_\zeta H_\varrho(z,\zeta) = -\varrho(\zeta)^2\,Q_\varrho(z,\zeta).$$

If the harmonic compensator is positive, then a local study of the behavior near $\mathbb{T} \times \mathbb{T}$ reveals that $Q_\varrho|_{\mathbb{T}^2 \setminus \delta(\mathbb{T})} \leq 0$, where $\delta(\mathbb{T}) = \{(z,z) : z \in \mathbb{T}\}$ denotes the diagonal. We are led to search for some kind of reverse implication. We first study the reproducing kernel function K_ϱ for the space $P^2(\mathbb{D}, \varrho)$ which is obtained as the closure of the (holomorphic) polynomials with respect to the norm of $L^2(\mathbb{D}, \varrho)$. It is shown that $Q_\varrho = 2\,\mathrm{Re}\,K_\varrho - 1$, so that the information obtained for K_ϱ can be readily converted to information about Q_ϱ. This identity reflects the fact that under the reproducing condition on ϱ, the analytic polynomials and the antiananlytic polynomials vanishing at the origin are perpendicular to each other in the Hilbert space $HP^2(\mathbb{D}, \varrho)$. We obtain a representation formula for K_ϱ,

$$K_\varrho(z,\zeta) = \frac{1 - z\bar{\zeta}L_\varrho(z,\zeta)}{(1 - z\bar{\zeta})^2}, \qquad (z,\zeta) \in \mathbb{D} \times \mathbb{D},$$

whereby L_ϱ is the reproducing kernel for some Hilbert space of analytic functions on $\mathbb{D}$, which may be called the *deficiency space* for $P^2(\mathbb{D}, \varrho)$. Using this representation, we find that that Q_ϱ is negative on $\mathbb{T}^2 \setminus \delta(\mathbb{T})$, and in fact that

$$Q_\varrho(z,\zeta) \leq -\left(\frac{1}{\varrho(z)^2} + \frac{1}{\varrho(\zeta)^2}\right) \frac{1}{|z - \zeta|^2}, \qquad (z,\zeta) \in \mathbb{T} \times \mathbb{T} \setminus \delta(\mathbb{T}), \tag{4.1}$$

and just as previously this allows us to go backwards, to obtain the positivity of the harmonic compensator, by means of a variational technique along the weighted Hele-Shaw flow. We use the Hadamard variational method for the laplacian to write the Green function G as a negative integral of a product of two Poisson kernels for the flow domains $D(r)$ over r, $0 < r < 1$. Noting that $H_\varrho(\cdot,\zeta)$ is the orthogonal harmonic projection (with respect to the weight) of the function $-G(\cdot,\zeta)$, we find that it suffices to show that the harmonic projection of a positive harmonic function on a flow region $D(r)$, with $0 < r < 1$, extended to vanish on $\mathbb{D} \setminus D(r)$, is positive throughout $\mathbb{D}$. This is precisely what the estimate (4.1) permits us to do.

References

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