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RECENT EXISTENCE AND REGULARITY RESULTS FOR
WAVE MAPS

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The setting. We consider maps $u$ from $(m+1)$-dimensional Minkowski space to a compact, $k$-dimensional Riemannian manifold $(N, g)$ with $\partial N = \emptyset$, the "target". By Nash's embedding theorem, we may assume that $N \subset \mathbb{R}^n$, isometrically, for some $n > k$. We denote as $T_pN \subset T_p\mathbb{R}^n \cong \mathbb{R}^n$ the tangent space of $N$ at a point $p$, and we denote as $T^\perp_pN$ the orthogonal complement of $T_pN$ with respect to the inner product $\langle \cdot, \cdot \rangle$ in $\mathbb{R}^n$. $TN, T^\perp N$ will denote, respectively, the corresponding tangent and normal bundles.

The space-time coordinates will be denoted as $z = (t, x) = (x^\alpha)_{0 \leq \alpha \leq m}$ and we denote as $\frac{\partial}{\partial x^\alpha}u = \partial_\alpha u = u_\alpha$ the partial derivative of $u$ with respect to $x^\alpha$, $0 \leq \alpha \leq m$. Also let $D = \left( \frac{\partial}{\partial t}, \nabla \right) = \left( \frac{\partial}{\partial t^\alpha}, \partial_\alpha \right)_{0 \leq \alpha \leq m}$ let $\eta$ be the Minkowski metric $\eta = (\eta_{\alpha\beta}) = (\eta^{\alpha\beta})^{-1} = \text{diag}(-1, 1, \ldots , 1)$. We raise and lower indices with the metric. By convention, we tacitly sum over repeated indices. Thus, for example, $\partial^\alpha = \eta^{\alpha\beta} \partial_\beta$. Moreover,

$$\Box = -\partial^\alpha \partial_\alpha = \frac{\partial^2}{\partial t^2} - \Delta$$

is the wave operator and

$$\frac{1}{2} \langle \partial^\alpha u, \partial_\alpha u \rangle = \frac{1}{2} (|\nabla u|^2 - |u_t|^2)$$

is the Lagrangean density of $u$.

Wave maps. A map $u$ is a wave map if $u$ is a stationary point for the action integral

$$A(u; Q) = \frac{1}{2} \int_Q \langle \partial^\alpha u, \partial_\alpha u \rangle \, dz$$

with respect to compactly supported variations $u_\varepsilon: \mathbb{R} \times \mathbb{R}^m \to N, |\varepsilon| < \varepsilon_0$, such that $u_\varepsilon = u$ outside a compact set in space-time and for $\varepsilon = 0$, in the sense that

$$\left. \frac{d}{d\varepsilon} A(u_\varepsilon; Q) \right|_{\varepsilon = 0} = 0$$

for any $Q \subset \mathbb{R} \times \mathbb{R}^m$ strictly containing the support of $u_\varepsilon - u$.

Wave maps then satisfy the relation

$$\Box u \perp T^\perp u.$$

To understand this relation in more explicit terms, fix a point $z_0 \in \mathbb{R} \times \mathbb{R}^m$ and let $\nu_{k+1}, \ldots , \nu_n$ be an orthonormal frame for $T^\perp_N$, smoothly depending on

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$p \in N$ for $p$ near $p_0 = u(z_0)$. Then we can find scalar functions $\lambda^l : \mathbb{R} \times \mathbb{R}^m \to \mathbb{R}$, $k < l \leq n$, such that near $z = z_0$ there holds

$$\Box u = \lambda^l (\nu_l \cdot u);$$

in fact,

$$\lambda^l = \langle \Box u, \nu_l \cdot u \rangle
= -\partial^a (\partial_a u, \nu_l \cdot u) + \langle \partial_a u, \partial^a (\nu_l \cdot u) \rangle
= \langle \partial_a u, d\nu_l (u) \cdot \partial^a u \rangle = A^l (u) (\partial_a u, \partial^a u)$$

is given by the second fundamental form $A^l$ of $N$ with respect to $\nu_l$. Thus, the wave map equation takes the form

$$\Box u = A(u) (\partial_a u, \partial^a u) \perp T_u N,$$

where $A = A^l \nu_l$ is the second fundamental form of $N$.

**Examples.** i) For $N = S^k \subset \mathbb{R}^{k+1}$ equation (0.1) translates into the particularly simple equation

$$\Box u = (|\nabla u|^2 - |u_t|^2) u.$$  

Indeed, since $u \perp T_u S^k$ it suffices to check that

$$\langle \Box u, u \rangle = -\partial^a (\partial_a u, u) + \langle \partial_a u, \partial^a u \rangle = |\nabla u|^2 - |u_t|^2.$$  

ii) Suppose $\gamma : \mathbb{R} \to N$ is a geodesic parametrized by arc-length and $u = \gamma \circ v$ for some map $v : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$. Compute

$$-\Box u = \partial^a (\gamma'(v) \partial_a v) = \gamma''(v) \partial_a v \partial^a v - \gamma'(v) \Box v.$$  

Note that $\gamma'$ is parallel along $\gamma$; that is, $\gamma''(s) \perp T_{\gamma(s)} N$ for all $s \in \mathbb{R}$. Thus, $u$ satisfies (0.1) if and only if $v$ solves the linear, homogeneous wave equation $\Box v = 0$.

**Basic questions.** In view of the hyperbolic nature of equation (0.1), it is natural to ask whether the Cauchy problem for equation (0.1) for (sufficiently) smooth initial data

$$(u, u_t) |_{t=0} = (u_0, u_1) : \mathbb{R}^m \to TN$$

always admits a unique smooth solution for small time $|t| \leq T$. That is, we consider data $u_0 : \mathbb{R}^m \to N, u_1 : \mathbb{R}^m \to \mathbb{R}^n$ such that $u_1(x) \in T_{u_0(x)} N$ for almost every $x \in \mathbb{R}^m$.

The smoothness hypothesis on the solution and the data may be rather weak. In fact, for a function $u \in L^2_{\text{loc}} (\mathbb{R} \times \mathbb{R}^m ; N)$ it is possible to interpret equation (0.1) in the sense of distributions provided $Du \in L^2_{\text{loc}} (\mathbb{R} \times \mathbb{R}^m)$. More generally, we may consider initial data $(u_0, u_1)$ in Sobolev spaces $H^s \times H^{s-1}(\mathbb{R}^m ; TN), s \geq 1$, and solutions $u$ of class $H^s$, that is, such that $(u, u_t) \in L^\infty (\mathbb{R} ; H^s \times H^{s-1}(\mathbb{R}^m ; TN))$.

Then we may ask for which $s$ the initial value problem (0.1), (0.2) with data $(u_0, u_1) \in H^s \times H^{s-1}(\mathbb{R}^m ; TN)$ admits a unique local solution of class $H^s$ ("local well-posedness in $H^s$") and for which $s$ this solution may be extended for all time and also preserves higher regularity properties of the data ("global well-posedness" and regularity).

A dimensional analysis tells us what we may hope for. Assigning scaling dimensions 1 to each coordinate $x^i$, 0 to the function $u$, the $H^s$-energy in $m$ space dimensions has dimension $m - 2s$; that is, if $s > \frac{m}{2}$, no concentration discontinuities
on length scales $L \to 0$ are possible if the $H^s$-energy of $u$ remains bounded. We refer to this case as sub-critical, in contrast to the critical and supercritical cases $s = \frac{m}{2}, s < \frac{m}{2}$, respectively.

By a fixed point argument, using only classical energy estimates (for $u$ and derivatives), for a general hyperbolic equation $\Box u = f(u, Du)$ with a smooth function $f$ it is not hard to establish local well-posedness of the Cauchy problem in $H^s$, if $s > \frac{m}{2} + 1$.

Using, however, the special geometric, analytic, and algebraic structure properties of the wave map system, this result can be improved drastically.

**Geometric structure.** Orthogonality $\Box u \perp T_u N$ immediately implies the conservation law

$$0 = \langle \Box u, u_t \rangle = \frac{1}{2} \frac{d}{dt} |Du|^2 - \text{div}(\nabla u, u_t).$$

Integrating over $\mathbb{R}^m$, if $Du(t)$ has spatially compact support, we obtain the energy identity

$$E(u(t)) := \frac{1}{2} ||Du(t)||_{L^2(\mathbb{R}^m)}^2 = \text{const.} \quad (0.3)$$

Similarly, we can argue for higher derivatives. Suppose $(u_0, u_1) \in H^2 \times H^1(\mathbb{R}^m; TN)$. Let $\partial$ be any first order spatial derivative. Differentiating equation (0.1), we obtain

$$\Box (\partial u) = \partial [A(u)(\partial u, \partial u)] = dA(u)(\partial u, \partial u, \partial^a u) + 2A(u)(\partial u, \partial u, \partial^a u)$$

with data

$$\langle \partial u, \partial u_t \rangle |_{t=0} = (\partial u_0, \partial u_1) \in H^1 \times L^2(\mathbb{R}^m; \mathbb{R}^n).$$

Note that, since $\langle u_1, A(u)(\cdot, \cdot) \rangle = 0$ by orthogonality, we have

$$\langle \partial u_1, A(u)(\partial u, \partial u, \partial^a u) \rangle = -\langle u_1, dA(u)(\partial u, \partial u, \partial^a u) \rangle.$$ 

Hence we obtain

$$\frac{d}{dt} E(\partial u(t)) = \int_{\{t\} \times \mathbb{R}^m} \langle \Box (\partial u), \partial u_t \rangle \, dx$$

$$\leq C ||dA(u)||_{L^\infty} \cdot \int_{\mathbb{R}^m} |Du(t)|^2 |D^2 u(t)| \, dx.$$ 

Since $N$ is compact, $dA$ is uniformly bounded on $N$. Moreover, by Sobolev's embedding, we can estimate

$$\int_{\mathbb{R}^m} |Du(t)|^2 |D^2 u(t)| \, dx \leq C ||Du(t)||_{L^2}^{\alpha-2} ||D^2 u(t)||_{L^2}^2,$$

where $\alpha = 2, 3, \text{ or } 4$ if $m = 1, 2, \text{ or } 3$, respectively.

Thus, by (0.3) we arrive at a Gronwall type inequality

$$\frac{d}{dt} ||D^2 u(t)||_{L^2}^2 \leq C ||D^2 u(t)||_{L^2}^2.$$ 

A local-in-time $H^2$-bound follows. If $m = 1$, we have $\alpha = 2$, and we even obtain global unique $H^2$-solutions. We summarize these facts in the following result.

**Theorem 0.1.** Suppose $m \leq 3$. Then for any data $(u_0, u_1) \in H^2 \times H^1(\mathbb{R}^m; TN)$ there exists a unique local solution $u$ of class $H^2$. If $m = 1$, the solution extends uniquely for all time. If $(u_0, u_1) \in H^s, s > 2$, then so is $u$. 

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For $m = 1$, the above result is due to Gu [11] and Ginibre-Velo [10]; in [17], Shatah gave a very elegant and concise proof. Finally, Yi Zhou [21] showed that the initial value problem is globally well-posed even in the energy space $H^1$.

For $m = 2, 3$ the above result also was obtained by Klainerman-Machedon [13] by a completely different technique. The above proof was first given in [20]; proof of Theorem 3.3. See also Choquet-Bruhat [2] for early results on wave maps.

**Analytic structure.** As illustrated best by the wave map system for maps to the sphere, equation (0.1) also exhibits the special analytic structure of “null forms” in the sense of Klainerman-Machedon [13].

As a simple model, consider solutions $u : \mathbb{R} \times \mathbb{R}^m \to \mathbb{R}$ of the equation

$$\Box u = |\nabla u|^2 - |u_t|^2 \text{ on } \mathbb{R} \times \mathbb{R}^m$$

(0.4)

with initial data $u_{t=0} = 0, u_{t=0} = u_1 \in H^{s-1} (\mathbb{R}^m)$.

Letting $v = e^u$, we compute

$$\Box v = e^u (\Box u - |\nabla u|^2 + |u_t|^2) = 0$$

with $v_{t=0} = 1, v_{t=0} = u_1 \in H^{s-1} (\mathbb{R}^m)$.

By exact dependence of the solution $v$ on its data in $H^s \times H^{s-1} (\mathbb{R}^m)$, we have $v \in C^0 (\mathbb{R}; H^s (\mathbb{R}^m))$. On the other hand, a necessary condition for $v$ to arise as $v = e^u$ from a (local) solution $u$ to (0.4) is $v > 0$ (for short time), which requires $H^s (\mathbb{R}^m) \hookrightarrow L^\infty (\mathbb{R}^m)$, that is, $s > \frac{m}{2}$.

In remarkable agreement with this classical example, Klainerman-Machedon [14] establish the following result.

**Theorem 0.2.** The initial value problem (1), (2) is locally well-posed for data $(u_0, u_1) \in H^s \times H^{s-1} (\mathbb{R}^m; TN)$ with $s > \frac{m}{2}$.

This result underscores the importance of the critical case $s = \frac{m}{2}$, in particular, the case $s = 1$ in $m = 2$ space dimensions. Progress on this issue can be made by taking into account a third structure property of the wave map system.

**Algebraic structure.** As an illustration, first consider the case of a homogeneous target space $N = G/H$, where $G$ is a Lie group and $H$ is a discrete subgroup of $G$.

Then there exist Killing vector fields $Y_i$ spanning $T_p N$ at any point $p \in N$ and (0.1) is equivalent to the system of equations

$$0 = \langle \Box u, Y_i \circ u \rangle = -\partial^\alpha \langle \partial_\alpha u, Y_i \circ u \rangle + \langle \partial_\alpha u, dY_i (u) \cdot \partial^\alpha u \rangle$$

for all $i$. Since $Y_i$ is Killing, the last term vanishes and we obtain the first order Hodge system

$$-\partial^\alpha \langle \partial_\alpha u, Y_i \circ u \rangle = 0$$

(0.5)

for all $i$, equivalent to (0.1). This form of (0.1) immediately implies the following weak compactness result. Suppose $(u^k)$ is a sequence of wave maps such that $u^k \to u$ in $L^2, Du^k \to Du$ weakly in $L^2$, locally, as $L \to \infty$. Then $u$ again is a (weak) wave map.
Coupling this observation with a suitable scheme for obtaining approximate solutions to (0.1), Shatah [17] (for \( N = S^k \)), Yi Zhou [22] (for \( m = 2 \)), and Freire [7] (for the general case) then obtain the following result.

**Theorem 0.3.** Suppose \( N = G/H \) is homogeneous. Then for any \((u_0, u_1) \in H^1 \times L^2(\mathbb{R}^m; TN)\) there exists a global weak solution \( u \) of (0.1), (0.2) of class \( H^1 \).

In the case of a general target manifold, the algebraic structure giving rise to a Hodge system analogous to (0.5) was uncovered independently by Christodoulou-Tahvildar-Zadeh [3] and Hélein [12]. With no loss of generality (as shown by these authors) we may assume that \( TN \) is parallelizable; that is, there exists a smooth orthonormal frame field \( \tilde{e}_1, \ldots, \tilde{e}_k \) for \( TN \). Given a (weak) wave map \( u : \mathbb{R} \times \mathbb{R}^m \to N \), we then obtain a frame for the pull-back bundle \( u^{-1}TN \) by letting

\[
e_i(z) = R_{ij}(z) \tilde{e}_j(u(z)) \quad \text{for } z = (t, x) \in \mathbb{R} \times \mathbb{R}^m,
\]

where

\[
R = (R_{ij}) : \mathbb{R} \times \mathbb{R}^m \to SO(k).
\]

Denote \( \theta_i = \langle du, e_i \rangle = \theta_{i,a} dx^a, \omega_{ij} = \langle de_i, e_j \rangle = \omega_{ij,a} dx^a, 1 \leq i, j \leq k. \)

Then (0.1) is equivalent to the system of equations

\[
0 = \Box u, e_i = -\partial^a (\partial_a u, e_i) + (\partial_a u, \partial^a e_i) = -\partial^a \theta_{i,a} + \omega_{ij}^a \cdot \theta_{j,a} = : \delta^a \theta_{i,a} + \omega_{ij} \cdot \theta_j
\]

for \( 1 \leq i \leq k. \) Note that (0.6) is a first order Hodge system analogous to (0.5); however, (0.6) differs from (0.5) by a quadratic expression.

Using the Hodge structure (0.6), in joint work with A. Freire and S. Müller [8], [9] we obtain weak compactness of wave maps in \( m = 2 \) space dimensions.

**Theorem 0.4.** Let \( m = 2. \) Suppose \( (u^L) \) is a sequence of wave maps such that \( u^L \to u \) in \( L^2 \) and \( Du^L \to Du \) weakly in \( L^2 \), locally on \( \mathbb{R} \times \mathbb{R}^m \), as \( L \to \infty \). Then \( u \) is a (weak) wave map.

The proof makes contact with the work of Evans [5] and Bethuel [1] on the partial regularity of stationary harmonic maps. In particular, we also use special compensation properties of Jacobians ([4]) and \( H^1 - BMO \) duality ([6]).

The crucial determinant structure for the nonlinear term in (0.6) is achieved by localizing the equation to a compact domain which we then regard as contained in the fundamental domain of a torus \( T^3 = \mathbb{R}^3/\mathbb{Z}^3. \)

On \( T^3 \) (following Hélein [12]) we then impose the Coulomb gauge condition (with respect to the Euclidean background metric) by choosing, for each \( L, \) a "gauge" \( R^L \in H^1(T^3, SO(k)) \) such that

\[
\sum_i \int_{T^3} |D e_i^L|^2 dz = \min_R \sum_i \int_{T^3} |D( R_{ij} \tilde{e}_j\cdot u^L)|^2 dz.
\]

In this gauge, we have

\[
\partial_a \omega_{ij,a} = \delta_{\text{Eucl}} \omega_{ij} = 0,
\]

and \( (e^L_i) \) is bounded in \( H^{1,2}(T^3) \) with

\[
\sum_i \int_Q |D e_i^L|^2 dz \leq \sum_i \int_Q |D( e_i\cdot u^L)|^2 dz \leq CE( u^L(0)) \leq C.
\]
Hence we may assume that \( e^L_i \to e_i \) weakly in \( H^{1,2}(T^3) \) and
\[
\begin{align*}
\theta^L_i &= \langle du^L, e^L_i \rangle = \theta^L_{i,\alpha} dx^\alpha \to \theta_i = \langle du, e_i \rangle, \\
\omega^L_{ij} &= \langle de^L_i, e^L_j \rangle = \omega^L_{ij,\alpha} dx^\alpha \to \omega_{ij} = \langle de_i, e_j \rangle
\end{align*}
\]
weakly in \( L^2 \) as \( L \to \infty \).

Using the Hodge \(*\)-operator (with respect to \( \eta \)), we may express
\[
\omega^L_{ij} \cdot \eta \theta^L_j \, dz = \omega^L_{ij} \wedge (\ast \eta \theta^L_j).
\]
By Hodge decomposition (with respect to the Euclidean metric on \( T^3 \)), moreover, we have
\[
\ast \eta \theta^L_j = da^L_j + \delta_{\text{euc}b^L_j + c^L_j,}
\]
where \( a^L_j \to a_j, b^L_j \to b_j, c^L_j \to c_j \) in \( H^1(T^3) \) as \( L \to \infty \). The harmonic forms \( c^L_j \)
are constant multiples of the basis vectors \( dx^\alpha \wedge dx^\beta \); hence \( c^L_j \to c_j \) smoothly, as
\( L \to \infty \), and \( \omega^L_{ij} \cdot \eta c^L_j \to \omega_{ij} \cdot c_j \) in \( \mathcal{D}' \). Using the Coulomb gauge condition, and
letting \( \beta^L_j = \ast \eta b^L_j \), the second term may be re-written
\[
\omega^L_{ij} \wedge \delta_{\text{euc}b^L_j = \delta_{\text{euc}}(\omega^L_{ij} \beta^L_j) \, dz,
\]
which tends to the desired distributional limit. Similarly, for the third term we have
\[
\omega^L_{ij} \wedge da^L_j = -d(\omega^L_{ij} \wedge a^L_j) + d\omega^L_{ij} \wedge a^L_j.
\]
Again, it is easy to pass to the limit \( L \to \infty \) in the divergence term. The last term, finally, possesses a determinant structure
\[
d\omega^L_{ij} \wedge a^L_j = de^L_i \wedge de^L_j \wedge a^L_j.
\]
Using the Hardy space estimates for Jacobians of \([4]\) and \( H^1 - BMO \) duality of \([6]\) we are able to show that, as \( L \to \infty \),
\[
de^L_i \wedge de^L_j \wedge a^L_j \to de_i \wedge de_j \wedge a_j + \nu \text{ in } \mathcal{D}',
\]
and to characterize the defect measure \( \nu \) in a way analogous to P.L. Lions' \([15]\) concentration-compactness principle. In particular, from energy estimates we derive that the \( H^1 \)-capacity of the support of \( \nu \) vanishes. But, passing to the limit \( L \to \infty \) in \((0.6)\), on the other hand we have
\[
0 = \delta_{\eta} \theta^L_i + \omega^L_{ij} \cdot \eta \theta^L_j \to \delta_{\eta} \theta_i + \omega_{ij} \cdot \eta \theta_j + \nu \text{ in } \mathcal{D}';
\]
that is,
\[
\nu = -\delta_{\eta} \theta_i - \omega_{ij} \cdot \eta \theta_j \in H^{-1} + L^1(T^3),
\]
and hence \( \nu = 0 \).

Finally, in joint work with S. Müller \([16]\) we couple the above weak compactness argument with the viscous approximation method suggested by Yi Zhou \([22]\) to obtain

**Theorem 0.5.** Let \( m = 2 \). Then for any \((u_0, u_1) \in H^1 \times L^2(\mathbb{R}^m;TN)\) there exists a global weak solution to the Cauchy problem \((0.1), (0.2)\).

It remains to question whether this solution is unique and regular for smooth data.
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