FRANK MERLE

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Recent Progress on the Blow-up Problem for Zakharov Equations

By Frank Merle

Université de Cergy-Pontoise
Mathématiques, 8 Avenue du Parc, Le Campus, 95033 Cergy-Pontoise Cedex France

In this paper, we present recent progress for the blow-up problem for Zakharov equations.

More precisely, we consider Zakharov equations

\[ \begin{align*}
  i\partial_t u - \Delta u + nu &= 0 \\
  \partial_t n - \nabla \cdot \n &= 0 \\
  c_0^{-2} \partial_t v &= -\nabla n - \nabla |u|^2
\end{align*} \]

with initial data \((u(-1), n(-1), v(-1)) = (u_0, n_0, v_0)\)

where \(u : \mathbb{R}^2 \to \mathbb{C}, n : \mathbb{R}^2 \to \mathbb{R}, v : \mathbb{R} \to \mathbb{R}^2\),

and related equations which are

the cubic nonlinear Schrödinger equation

\[ \begin{align*}
  i\partial_t u - \Delta u - |u|^2 u &= 0
\end{align*} \]

with initial data \(u(-1) = u_0\)

where \(u : \mathbb{R}^2 \to \mathbb{C}\),

and the Elliptic equation associated with (I)

\[ \begin{align*}
  u &= \Delta u + |u|^2 u
\end{align*} \]

where \(u : \mathbb{R}^2 \to \mathbb{C}\).

1) The local Cauchy theory for equations (I),(II).

We are interested to find a space \(H\) for equation (I) or (II) such that there is a unique solution of the equation on \([0,T)\) and we have the following \(T=+\infty\) or \(T<+\infty\) and \(|u(t)|_H \to +\infty\) as \(t\) goes to \(T\).

i)Case of the nonlinear Schrödinger equation (II)

The case of the cubic nonlinear Schrödinger equation is now well-understood. A local (in time) Cauchy theory can be done in various natural space \(H^1, H^S, L^2\) (see [GV],[K],[CaW],[Bo1]).

Moreover, one can show that the blow-up time does not depend on the Cauchy space and in fact
we have at the blow-up a concentration phenomenon in $L^2$.

In [MT] (see also [W2], [G1M2]), it is proved the following. Let $u(t)$ a blow-up solution (and $T$ its blow-up time), there is then $x(t)$ such that for all $R>0$, $\liminf_{t \to T} |u(t)|^2_{L^2(x-x(t) \leq R)} \geq a > 0$ where $a$ is an universal constant ($a=|Q|^2_{L^2}$ where $Q$ will be defined in subsection 3).

In addition, we have the following conserved quantities for all $t$,

$|u(t)|^2_{L^2} = |u_0|^2_{L^2}$,

$E(u(t)) = E(u_0)$ where $E(u) = 1/2|\nabla u|^2 - 1/4|u|^4$.

ii) Case of Zakharov equations (I).

A local (in time) Cauchy theory can not be done up to now in the energy space $H_1=\{(u,n,v)\in H^1 \times L^2 \times L^2\}$ for a general initial data. The result is proved for the space $H_2=\{(u,n,v)\in H^2 \times H^1 \times H^1\}$ (see for example [OT2], [KePVg], [Bo2] and the references therein).

Moreover, one can show that we have at the blow-up time again the same concentration phenomenon in $L^2$. Indeed, let $(u(t),n(t),v(t))$ a blow-up solution (and $T$ its blow-up time), there is then $x(t)$ such that for all $R>0$, $\liminf_{t \to T} |u(t)|^2_{L^2(x-x(t) \leq R)} \geq |Q|^2_{L^2}$ where $Q$ will be defined in subsection 3.

In addition, we have the following conserved quantities for all $t$,

$|u(t)|^2_{L^2} = |u_0|^2_{L^2}$,

$H(u(t),n(t),v(t)) = H(u_0,n_0,v_0)$ where $H(u) = |\nabla u|^2 + n|u|^2 + n^2/2\lambda + |v|^2/2c_0$.

iii) Blow-up problem

We are now interested in the case $T<\infty$, that is the case of a blow-up solution (or equivalently a singular solution) for equation (I) or (II). Most of the results can be extend in dimension $N=1$ in the case of a critical power for the nonlinear Schrödinger equation. Part of the results for the Zakharov equation can be extend to the dimension 3 (only dimensions 2,3 are relevant).

2) Elementary relations between equations (I)-(II)-(III)

i) Limit as $c_0$ goes to infinity.

We can easily see that as $c_0$ goes to infinity, the wave part of equation (I) give formally

$\nabla(n + |u|^2) = 0$,

or equivalently

$n + |u|^2 = 0$.

Thus equation (I) transform in equation (II) as $c_0$ goes to infinity.

If the initial data are compatible, this result of convergence has been rigourously proved by several authors ([AA2], [OT1], [KePVe]) when the limit solution $u(t)$ (of equation (II)) is regular. Near the blow-up time, we do not have convergence results and in some sense we can not expect some. For example, in [G1M2], there is the case of a blow-up solution of equation (II) with initial data $u_0$ such that for all finite $c_0$ and all $n_0,v_0$ the solution of (I) $(u,n,v)(t)$ is globally defined in time. Therefore, in some sense at the singularity, equation (I) when $c_0$ is large, can
not be consider as a perturbation of equation (II).

ii) Periodic solutions of (I),(II)

By direct calculation, we can check that if \( w(x) \) is a solution of equation (III) then

- \( u(t,x) = e^{it} \ w(x) \) is a periodic solution of equation (II)
- \( (u(t,x),n(t,x),v(t,x)) = (e^{it} \ w(x),-l \ w(x)) \) is also a periodic solution of equation (I).

iii) Conformally self-similar blowing-up solution

For this power in two dimension, the nonlinear Schrödinger equation has one more invariance: if \( u(t,x) \) is a solution of equation (II) then

\[
1/\tau(t,x) \exp(\frac{i \mu^2}{4t})
\]

is also a solution of equation (II).
In particular, if \( w(x) \) is a real solution of the equation (III), then

\[
1/t \ w(x/t) \exp(-i/t + i\mu^2/4t)
\]

is also a solution of equation (II) which blow-up at \( T = 0 \). We then obtain explicit blow-up solutions of equation (II).

Unfortunately, such invariance does not exist for the Zakharov equation. In particular, there is no direct way to obtain explicit blow-up solutions of Zakharov equations.

3) On minimal solutions of (III)

In this section, we recall briefly some results on the elliptic equation (III). From [BeL],[St] it is now classical that equation (III) have infinitly many solutions in \( H^1 \) (up to the invariance of the equation).

Let us defined the unique positive radially symmetric solution of equation (III) (see [Kw] for uniqueness). We have in fact that the solution \( w=0 \) is isolated in the set of solution in \( L^2 \). More precisely,

i) Assume that \( w(x) \) is a nonzero solution of equation (III) then \( |w|_L^2 \leq |Q|_L^2 \).

ii) Moreover, we have the following caracterisation of the minimal solution (or ground state) of equation (III). Assume that \( w \) is a nonzero solution of equation (III) and \( |w|_L^2 = |Q|_L^2 \) then up to the invariance of the equation \( w = Q \) (that is there exist \( x',\omega,\theta \) such that \( w(x) = e^{i \theta} \omega Q(\omega(x-x')) \)).

4) Equation (II)

The problem of singularity for equation (II) has been studied in the last 20 years, and we give here part of results obtained.

i) No blow-up for small data

In [W1], it has been proved that for \( u \in H^1 \), we have the following

\[
\frac{1}{4} \int |u|^4 \leq \frac{1}{2} \int \nabla^2 (|u|^2 / |Q|^2).
\]

It follows from this identity that if

\[
|u_0|_L^2 < |Q|_L^2
\]

then there is non blow-up phenomenon and the solution is globally defined in time.

ii) blow-up for large data

For this equation there are two way to obtain blow-up solutions.

- explicit blow-up solution.
From the conformal invariance of the equation if \( w(x) \) is a real solution of the equation (III),
then
\[ \frac{1}{t} w(x/t) \exp(-i/t + ilx^2/4t) \]
is also a solution of equation (II) which blow-up at \( T = 0 \).
In particular
\[ S(t,x) = \frac{1}{t} Q(x/t) \exp(-i/t + ilx^2/4t) \]
is a blow-up solution such that \( \| u_0 \|_{L^2} = \| Q \|_{L^2} \).

- Viriel identity.
From [SoSyZ], [Gla], we have the following property of the solution of equation (II).
Assume that \( |x| u_0 \in L^2 \) then for all time \( t \), \( |x| u_0 \in L^2 \) and
\[
d^2/dt^2 \left\{ \int |x|^2 |u(t,x)|^2 \, dx \right\} = 16 \, E(u_0).
\]
From this viriel identity, we have that
if \( E(u_0) < 0 \) then the solution blow-up in finite time (\( T < +\infty \)).

iii) Minimal blow-up solutions
Since if \( \| u_0 \|_{L^2} < \| Q \|_{L^2} \) then there is non blow-up, and there is blow-up solution in the case where \( \| u_0 \|_{L^2} = \| Q \|_{L^2} \), one can ask is it possible to characterize all minimal blow-up solutions in \( L^2 \) (that is solution which blows-up and such that \( \| u_0 \|_{L^2} = \| Q \|_{L^2} \)).
In [M1] (see also [M4] for another approach of the proof), the following is proved.
Assume that \( u(t) \) is a blow-up solution with minimal mass (and \( u(t) \) is an \( H^1 \) solution of equation (II)), that is \( \| u(t) \|_{L^2} = \| Q \|_{L^2} \). Then up to the invariance of the equation, we have
\[ u(t,x) = S(t,x) = \frac{1}{t} Q(x/t) \exp(-i/t + ilx^2/4t) \]
(that is there exist \( x', x'', \omega, \theta \) such that \( u(t,x) = e^{i\omega t} Q((x-x')/\omega t - x'\theta) \exp(-i\omega^2/4t + ilx-x'^2/4t) \).

5) Equation (I)

Until recently, there were no results on existence of solutions which blow-up for Zakharov equations. Indeed the two ingredients; the conformal invariance and the viriel identity which give blow-up results for the limit equation as \( \epsilon_0 \) goes to infinity do not hold. We can note that there were numerical evidence of singular behavior of solution of equation (I) in [LPSSW] and [PSSW].

i) No blow-up for small data
One can show (see [AA1], [SS]) as for the Schrödinger equation, that if \( \| u_0 \|_{L^2} < \| Q \|_{L^2} \) then there is non blow-up phenomenon and the solution is globally defined in time.

ii) Blow-up for large data
As for equation (II), we are able to construct explicit blow-up solution and give obstructions to regular behavior.

- Explicit blow-up solutions.
We do not have anymore the conformal invariance to obtain explicit blow-up solutions. We use in fact a bifurcation argument at "infinity" (using the structure of the nonlinear Schrödinger equation) to obtain explicit blow-up solution.
In [GIM1], a family of blow-up solutions in the energy space of the form
\[ u(t,x) = \omega t \, P(\omega x/t) \exp(-\omega^2/4t + ilx^2/4t) \]
\[ n(t,x) = \{ \omega t \}^2 \, N(\omega x/t) \]
where \( P(x) = P(|x|) \) and \( N(x) = N(|x|) \)
and
\[
P = \Delta P + NP
\]
\[
(c_0 \omega)^2 \left\{ r^2 N_{tt} + 6r N_t + 6N \right\} - \Delta N = \Delta P^2
\]
is investigated.
More precisely, it is proved using this kind of construction, that there are blow-up solutions such that \( u_0 \| x \|^2 = \| Q \| x \|^2 + \varepsilon \), for all \( \varepsilon > 0 \).

We can note that the solutions constructed are numerically stable (see [LPSSW]). The problem now is the following, we have construct blow-up solutions but we do not existance of many (or a large set) of singular solution. For this purpose, we use a different approach.

-viriel identity.

In [M2], it is derived a perturbed viriel identity for the Zakharov equation. More precisely, for a regular solution with decay at infinity we have
\[
d^2/\|x\|^2 \frac{\partial}{\partial t} \left\{ 1/4 \int |x|^2 u(t,x) \| x \|^2 \; dx + c_0^{-2} \int \int (u(t,x))(y(t,x)) \; dx \; dt \right\} = 2H(u_0,n_0,v_0) - c_0^{-2} \int \|v(t,x)\|^2 \; dx.
\]

From this perturbed viriel identity, we have in [M2] that
\[
\text{if } H(u_0,n_0,v_0) < 0 \text{ and the initial data are radially symmetric then the solution blow-up in finite time } (T<\infty) \text{ or in infinite time in } H^1 \text{ with a concentration of } u(t) \text{ in } L^2 \text{ as } t \to \infty.
\]
We suspect that in the case where \( H(u_0,n_0,v_0) < 0 \) then the solution away blows-up in finite time. This result give in particular the existence of a large class of singular solutions.

iii) Minimal blow-up solutions

Since if \( \| u_0 \| H^1 > \| Q \| H^1 \) then there is non blow-up, and there are blow-up solution such that \( \| u_0 \| H^1 = \| Q \| H^1 + \varepsilon \), for all \( \varepsilon > 0 \). one can ask, as for the nonlinear Schrödinger equation about minimal blow-up solutions (that is solution which blows-up and such that \( \| u_0 \| H^1 = \| Q \| H^1 \)).

In [G1M2], we in fact proved that there is no minimal blow-up solution:
\[
\text{if } \| u_0 \| H^1 = \| Q \| H^1 \text{ then there is non blow-up.}
\]
Therefore, the situation is different from the one of the nonlinear Schrödinger equation.

iv) Instability and stability results of blow-up behavior

Let us first recall some results for the nonlinear Schrödinger equation. We have explicit blow-up solution such that the blow-up rate in \( H^1 \) is of the type \( 1/(T-t) \). In particular, the one which the minimal blow-up solution has this rate of blow-up. From a physical point of view, we can expect that this rate is stable. It is not the case. Indeed, in [LPSS] for example it is observed numerically blow-up rate of the type \( \log \log (T-t)/ (T-t)^{1/2} \).

In [M3], it show for the Zakharov equation (with \( c_0 \) finite but eventually very large), that the blow-up rate is stonger than \( 1/(T-t) \). More precisely, let \( u,v(t,v)(t) \) a blow-up solution and \( T \) its blow-up time, we have for \( t \) near
\[
\| u(t) \| H^1 \geq c/(T-t)
\]
This shows that in fact the blow-up rate of the type \( \log 1/ (T-t)^{1/2} \) is unstable with respect to perturbations of the equation (with a term involving a wave equation). In contrary, the one with blow-up rate \( 1/(T-t) \) seem numerically stable. This in particular shows the physical interest of the minimal blow-up solution of the nonlinear Schrödinger equation : the solution of the form
\[
S(t,x) = 1/t \cdot Q(x/t) \exp(-i/t + ilx^2/4t).
\]

XX.5
References:


