ANTONIO SA BARRETO

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<http://www.numdam.org/item?id=JEDP_1991____A12_0>
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1 Introduction

Let \( \Omega \subset \mathbb{R}^3 \) be an open subset and let \( P \) be a second order strictly hyperbolic differential operator in \( \Omega \) with smooth coefficients. Let \( t \in C^\infty(\Omega) \) be a time function for \( P \) and define

\[
\Omega^\pm = \Omega \cap \{ \pm t > 0 \}.
\]  

(1.1)

Assume that \( \Omega \) is a domain of dependence of \( \Omega^- \). Let \( f \) be a smooth function of its arguments and suppose \( u, Du \in L^\infty_{\text{loc}}(\Omega) \) satisfies

\[
Pu = f(z, u, Du); \ z \in \Omega.
\]  

(1.2)

The general question on propagation of singularities of solutions of (1.1) is how do singularities of \( u \) in \( \Omega^- \) influence singularities of \( u \) in \( \Omega \). We shall concentrate in the study of some geometric singularities called conormal and the first example is conormality to a smooth hypersurface. Thus let \( S \subset \Omega \) be a smooth hypersurface which is characteristic for \( P \), let \( \mathcal{V}_S \) be the Lie algebra of smooth vector fields tangent to \( S \) and denote

\[
I_k L^2_{\text{loc}}(\Omega, \mathcal{V}_S) = \{ u \in L^2_{\text{loc}}(\Omega) : \mathcal{V}^j_S u \subset L^2_{\text{loc}}(\Omega), \ j \leq k \}. 
\]  

(1.3)

Observe that if \( u \in I_\infty L^2_{\text{loc}}(\Omega, \mathcal{V}_S) \), then \( u \) is smooth away from \( S \). In fact one can easily show that in this case the wavefront set of \( u \) is contained in the conormal bundle to \( S \).

Theorem 1.1 (Bony, [4]) Let \( u, Du \in H^s_{\text{loc}}(\Omega), \ s > \frac{3}{2}, \) satisfy (1.2). If \( u, Du \in I_k L^2_{\text{loc}}(\Omega^-, \mathcal{V}_S) \), then \( u, Du \in I_k L^2_{\text{loc}}(\Omega, \mathcal{V}_S) \).

This result shows that as long as \( S \) is smooth \( u \) remains conormal to it, but in general characteristic hypersurfaces of \( P \) can have rather complicated singularities. In this talk we shall describe the results of [16] and [17] concerning the propagation of conormal singularities for solutions of (1.2) along a hypersurface \( \Sigma \) with either a cusp or a swallowtail singularity. These are in some sense, see [2], the only cases where the singularities are stable under small perturbations. These problems have been also studied by M. Beals [3] and R. Melrose [9], in the case of the cusp and G. Lebeau, [6], [7] and J-M.
Delort [5] in the case of the swallowtail with the hypotheses that $P$ has real analytic coefficients and the regular part of $\Sigma$ is real analytic.

Before stating our results we have to introduce some notation. Let $\mathcal{W}$ be a Lie algebra and $C^\infty$ module of smooth vector fields on a manifold with corners $X$ and let $\mu$ be a smooth measure on $X$. The space of iteratively regular distributions with respect to $\mathcal{W}$ is then defined as

$$I_k L^2_{\mu,c}(X, \mathcal{W}) = \{u \in L^2_{\mu,c}(X); \mathcal{W}^j u \in L^2_{\mu,c}(X), \ j \leq k\}. \quad (1.4)$$

2 The Cusp

Let $G$ be a hypersurface with a cusp singularity at a line $L$, i.e there are local coordinates near $q \in L$ such that

$$G = \{(x, y, z) \in \Omega : y^3 = x^2\}, \quad L = \{(x, y, z) : x = y = 0\}. \quad (2.1)$$

Assume that the smooth part of $G$ is characteristic for $P$. Let $\mathcal{V}_G$ be Lie algebra of smooth vector fields tangent to $G$. It is easy to show that the Lie algebra $\mathcal{V}_G$ is characteristic complete, i.e

$$[P, \mathcal{V}_G] \subset \Psi^0(\Omega) \cdot P + \Psi^1(\Omega) \cdot \mathcal{V}_G + \Psi^1(\Omega). \quad (2.2)$$

Where $\Psi^j(\Omega)$ denotes the space of properly supported pseudodifferential operators of order $j$ in $\Omega$. Then by commutator methods, see [4], one obtains

**Theorem 2.1** Let $u, Du \in H^s_{loc}(\Omega)$, $s > \frac{3}{2}$, satisfy equation (1.2). If $u, Du \in I_k L^2_{loc}(\Omega, \mathcal{V}_G)$, then $u, Du \in I_k L^2_{loc}(\Omega, \mathcal{V}_G)$.

Next we recall the spaces of marked Lagrangian distributions introduced by R. Melrose in [9]. Let $\Lambda_G = \text{clos}[N^*(G \setminus L)]$, $\Lambda_G$ is a smooth conic Lagrangian submanifold of $T^*\mathbb{R}^3$. Let $\Lambda_L = N^*L$ and

$$\mathcal{M}_1(G) = \{A \in \Psi^1(\Omega) : a = \sigma_1(A) = 0 \text{ at } \Lambda_G,\quad H_a \text{ is tangent to } \Lambda_G \cap \Lambda_L\}. \quad (2.3)$$

$$\mathcal{M}_1(L) = \{A \in \Psi^1(\Omega) : a = \sigma_1(A) = 0 \text{ at } \Lambda_L,\quad H_a \text{ is tangent to } \Lambda_G \cap \Lambda_L\}. \quad (2.4)$$

Let

$$J^G_m(\Omega) = I_k L^2_{loc}(\Omega, \mathcal{M}_1(G)) + I_k L^2_{loc}(\Omega, \mathcal{M}_1(L)). \quad (2.5)$$

In [9] Melrose proves that

$$J^G_m \subset I_k L^2_{loc}(\Omega, \mathcal{V}_G) \quad (2.6)$$

and
Theorem 2.2 (Melrose, [9]) Let $u, Du \in H^s_{\text{loc}}(\Omega), s > \frac{3}{2},$ satisfy equation (1.2). If $u, Du \in J^{G,m}_k(\Omega^-)$, then $u, Du \in J^{G,m}_k(\Omega)$.

Finally we introduce a third space of distributions associated to the cusp. Observe that in local coordinates where (2.1) holds one finds that $G$ is invariant under the $\mathbb{R}^+$ action

$$m_3^{3,2}(x,y) = (s^3x, s^2y).$$

This leads to the definition quasi-homogeneous polar coordinates, thus consider the non-round circle

$$S^1_{3-2} = \{(\omega_1, \omega_2) \in \mathbb{R}^2 : \omega_1^4 + \omega_2^6 = 1\}$$

and the manifold with boundary

$$X_{3-2} = S^1_{3-2} \times [0, \infty) \times \mathbb{R}.$$ (2.9)

Then define the blow-down map

$$\beta_{3-2} : X_{3-2} \to \mathbb{R}^3, \quad \beta_{3-2}(\omega, r, z) = (r^3\omega_1, r^2\omega_2, z).$$ (2.10)

Let $W_G$ be the Lie algebra of smooth vector fields in $X_{3-2}$ which are tangent to $\partial X_{3-2}$ and to $\mathcal{G}^{(1)} = \text{clos} \beta_{3-2}^{-1}[G \setminus L]$. Let $\mu$ be the pull back of the Lebesgue measure by the map $\beta_{3-2}$. Then one defines

$$J^{G}_k(\Omega) = \{u \in \tilde{L}^2_{\text{loc}}(\Omega) : \beta_{3-2}^* u \in \tilde{I}_k L^2_c(X_{3-2}, W_G)\}.$$ (2.11)

One can easily show that the space $J^{G}_k(\Omega)$ does not depend on the choice of coordinates such that (2.1) holds. Then see [16], one can show that if $W_G^1$ is the Lie algebra of smooth vector fields in $X_{3-2}$ that are tangent to $\partial X_{3-2}$ to $\mathcal{G}^{(1)}$ and to the lines $\{\omega_1 = 0, r = 0\}, \{\omega_2 = 0, r = 0\}$, then the blow down map $\beta_{3-2}$ induces an isomorphism

$$\beta_{3-2}^* : J^{G,m}_k(\Omega) \leftrightarrow \tilde{I}_k L^2_c(X_{3-2}, W_G^1).$$ (2.12)

Similarly if $W_G^0$ is the Lie algebra of smooth vector fields that are tangent to $\mathcal{G}^{(1)}$ and vanish on $\partial X_{3-2}$, then

$$\beta_{3-2}^* : I_k L^2_c(\Omega, \mathcal{V}_G) \leftrightarrow I_k L^2_c(X_{3-2}, W_G^0).$$ (2.13)

In particular one obtains from (2.12) and (2.13) that

$$J^{G}_k(\Omega) \cong J^{G,m}_k \cong I_k L^2_{\text{loc}}(\Omega, \mathcal{V}_G).$$ (2.14)

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The main difficulty in proving a propagation theorem for $J^G_k(\Omega)$ is that this space is not known to have a microlocal characterization. One of the main results of [16] is the following elliptic regularity type of theorem

**Theorem 2.3** If $u, Du \in H^p_{loc}(\Omega) \cap I_k L^2_{loc}(\Omega, G)$ satisfies equation (1.2), then $u, Du \in J^G_k(\Omega)$.

Theorem 2.3 illustrates an important idea that will be used in the proof of Theorem 7.1. One first proves a propagation theorem for a bigger space which has a microlocal characterization and then uses the equation to show that the solution is actually in the smaller space.

### 3 The Swallowtail

Since the results we wish to prove are local we shall assume that $\Omega \subset \mathbb{R}^3$ is a sufficiently small neighborhood of $O = (0,0,0)$. Let $\Sigma \subset \Omega$ be a hypersurface with a swallowtail singularity at $O \in \Omega$, i.e there are smooth coordinates $(x,y,z)$ in $\Omega$ such that

$$\Sigma = \{(x,y,z) : \delta(\lambda) = \lambda^4 + z\lambda^2 + y\lambda + x = 0, \text{ has a double real root}\}. \quad (3.1)$$

$\Sigma$ has a cusp singularity at

$$L = \{(x,y,z) : x = -\frac{z^2}{12}, \ y^2 = (\frac{2}{3}z)^3\} \quad (3.2)$$

and a self-intersection at

$$H = \{(x,y,z) : y = 0, \ x = -\frac{z^2}{4}, \ z \leq 0\}. \quad (3.3)$$
The continuation of the line $H$ to values of $z > 0$ corresponds to the set of $(x, y, z)$ such that $\delta(\lambda)$ has two double complex roots and therefore is not included in $\Sigma$. Let $\Sigma_{\text{reg}} = \Sigma \setminus [L \cup H]$ be the regular part of $\Sigma$.

The discriminant of the polynomial $\delta(\cdot)$ is given by

$$\Psi(x, y, z) = 16xz^4 - 4y^2z^3 - 128x^2z^2 + 144xz^2y + 256x^3 - 27y^4.$$ (3.4)

Hence one deduces from (3.2) and (3.3) that

$$\Sigma_{\text{reg}} = \{(x, y, z) : \Psi(x, y, z) = 0, \ y \neq 0, \ x \neq \frac{z^2}{12}\}.$$ (3.5)

Assume that $\Sigma_{\text{reg}}$ is characteristic for $P$, i.e. if $p = \sigma^2(P)$ is its principal symbol,

$$p(d\Psi) = 0 \text{ at } \Sigma_{\text{reg}}.$$ (3.6)

Assume that $t(0) = 0$ and that

$$\Sigma^- = \Sigma \cap \Omega^-$$ (3.7)

is a smooth hypersurface of $\Omega^-$.

Let $Q$ be the light cone for $P$ over $O$, then, see Proposition 3.3, $Q \cap \Sigma = E \cup B$, where away from $O$, $\Sigma$ and $Q$ intersect transversally at $E$ and are tangent to third order along $B$. Let $\mathcal{V}(\Sigma)$ and $\mathcal{V}(\Sigma, Q)$ be the Lie algebras of smooth vector fields tangent to $\Sigma$ and to $\Sigma$ and $Q$ respectively.

The following is then a simple consequence of the results of [17].

**Theorem 3.1** Let $u, Du \in H^s_{\text{loc}}(\Omega), s > \frac{3}{2}$, satisfy (1.2). If $u, Du \in I_kL^2_{\text{loc}}(\Omega^-, \mathcal{V}(\Sigma, Q))$, then $u, Du \in I_kL^2_{\text{loc}}(\Omega, \mathcal{V}(\Sigma, Q))$.

One deduces from Theorem 3.1

**Theorem 3.2** Let $u, Du \in H^s_{\text{loc}}(\Omega), s > \frac{3}{2}$, satisfy (1.2). If $u, Du \in I_kL^2_{\text{loc}}(\Omega^-, \mathcal{V}(\Sigma))$, then $u, Du \in I_kL^2_{\text{loc}}(\Omega, \mathcal{V}(\Sigma, Q))$. 

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In fact the results of [17] are stronger, we show that under the hypotheses of Theorem 3.1 the solution is strongly conormal in the sense of Melrose and Ritter, [12], along $B$ and in the sense of [16] along the cusp line $L$ of $\Sigma$.

In this note we shall restrict ourselves to the case where $u$ satisfies the weakly semilinear equation

$$Pu = f(z, u), \quad z \in \Omega. \quad (3.8)$$

Since it contains all new ideas involved in the proof of Theorem 3.1

I would like to acknowledge that the main new ideas in [17], originated in joint works (in progress) with R.B. Melrose, [13], and with R.B. Melrose and M. Zworski, [14]. I would like to thank them for sharing their ideas with me, for their interest and encouragement. Possible errors in this manuscript are of course my own fault.

4 Outline Of The Proof

To prove Theorem 3.1 in the case of the weakly semilinear equation (3.6) we shall introduce a family of spaces $\mathcal{J}_k(\Omega) \subset I_k L^2_{loc}(\Omega, \mathcal{V}(\Sigma)), \ k \in \mathbb{N}_0$, satisfying the following properties:

1) $\mathcal{J}_{k+1}(\Omega) \subset \mathcal{J}_k(\Omega) \subset I^2_{loc}(\Omega), J_0(\Omega) = L^2_{loc}(\Omega)$.
2) $\mathcal{J}_k(\Omega)$ is a $C^\infty(\Omega)$-module.
3) $\mathcal{J}_k(\Omega) \cap L^\infty_{loc}(\Omega)$ is a $C^\infty$ algebra.
4) $u, Du \in \mathcal{J}_k(\Omega) \implies u \in \mathcal{J}_{k+1}(\Omega)$.
5) $Pu = f \in \mathcal{J}_k(\Omega), \ u = f = 0$ in $\Omega_T = \Omega \cap \{t < T\}$, then $u, Du \in \mathcal{J}_k(\Omega)$.
6) If $u, Du \in I_k L^2_{loc}(\Omega^-, \mathcal{V}(\Sigma))$ in $\Omega^-$ satisfy (3.8), then $u, Du \in \mathcal{J}_k(\Omega^-)$.

Proof of Theorem 3.1 : Suppose that such a family of spaces $\mathcal{J}_k(\Omega)$ has been constructed. We then proceed by an induction argument. Let $\chi \in C^\infty(\mathbb{R}), \ \chi(s) = 0, \ s < -\frac{1}{2}, \ \chi(s) = 1, \ s > 0$. We obtain from (1.8)

$$P\chi u = \chi f(z, u) + [P, \chi]u. \quad (4.1)$$

If $u, Du \in J_0(\Omega) \cap J_1(\Omega^-)$, it follows from properties 2, 3 and 4 that the right hand side of (4.1) is in $J_1(\Omega)$. Thus one deduces from property 5 that $u, Du \in J_1(\Omega)$. By the same argument it follows that if $u, Du \in J_\ell(\Omega) \cap J_{\ell+1}(\Omega^-), \ \ell < k$, then $u, Du \in J_{\ell+1}(\Omega)$. \hfill \Box

To define the spaces $\mathcal{J}_k(\Omega)$ we shall introduce a blow-down map

$$\beta : X \rightarrow \mathbb{R}^3 \quad (4.2)$$

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from a manifold with corners $X$ to $\mathbb{R}^3$ such that the lifts of $\Sigma$ and $Q$ by $\beta$ intersect each other and the boundary of $X$ transversally. We then define

$$J_k(\Omega) = \{ u \in L^2_{\text{loc}}(\Omega) : \mathcal{W}^j \beta^* u \in L^2_\mu(X), \ j \leq k \}. \quad (4.3)$$

Where $\mathcal{W}$ is a Lie algebra and $C^\infty(X)$ module of smooth vector fields in $X$ and $\mu$ is the lift of the Lebesgue measure of $\mathbb{R}^3$ under $\beta$. It will be a clear consequence of the definition of $X$ and $\mathcal{W}$ that $J_k(\Omega)$, defined by (4.3), satisfies properties 1, 2 and 4. It is a simple consequence of the Gagliardo-Nirenberg type of estimates of [11] that the spaces defined by (4.3) also satisfy property 3. Property 6 follows from Theorem 2.3 and from the results of [15]. The proof of property 5 is of course the most difficult one. The manifold with corners $X$ and the algebra $\mathcal{W}$ will be constructed in Section 6.

5 Model Case

An easy computation shows that, in coordinates where (3.3) holds, $\Sigma$ is invariant under the $\mathbb{R}^4$ action

$$m^{4-3-2}(x, y, z, t) = (s^4 x, s^3 y, s^2 z, t), \ s \in \mathbb{R}^4. \quad (5.1)$$

Let

$$M_r^{4-3-2}(\Omega) = \{ u \in C^\infty(\Omega) : \partial_x^a \partial_y^b \partial_z^c u(0, 0, 0, t) = 0, \ \forall a, b, c \in \mathbb{N}, \ 4a + 3b + 2c \leq r \} \quad (5.2)$$

be the ideal of smooth functions having Taylor series at

$$O = \{(x, y, z, t) \in \Omega; \ x = y = z = 0\}$$

consisting of terms of homogeneity $r$ or greater with respect to (5.1). A differential operator $P$ is said to have only terms of homogeneity $r'$ or greater, with respect to (5.1), if

$$P : M_r^{4-3-2} \rightarrow M_{r+r'}^{4-3-2}, \ r \in \mathbb{N}_0, \ r + r' \geq 0. \quad (5.3)$$

Simple computations show that if $P_0 = D_y^2 - D_x D_z$, then $\Sigma_{\text{reg}}$ is characteristic for $P_0$, in general one can prove, see [17] that

**Proposition 5.1** If $P$ and $\Sigma$ are as above and $(x, y, z, t)$ are smooth coordinates in which (3.3) holds, then

$$P = a(D_y^2 - D_x D_z) + P_{-5}, \ a \in C^\infty(\Omega), \ |a| > 0. \quad (5.4)$$

where $P_{-5}$ has only terms of homogeneity $-5$ or greater with respect to (5.1).
Let $Q_0$ be the light cone for $P_0$ over $O$, then one easily finds that
\[ Q_0 = \{(x,y,z) \in \Omega : y^2 - 4xz = 0\}. \] (5.5)
In this model we find that away from $O$, $Q_0$ and $\Sigma$ are tangent to third order along $B_0$ and intersect transversally along $E_0$, where
\[ B_0 = \{(x,y,z) \in \Omega : x = y = 0\}, \] (5.6)
\[ E_0 = \{(x,y,z) \in \Omega : x = \frac{3}{16} z^2, \ y^2 = -\frac{27}{32} z^3\}. \] (5.7)

Fig 3:

As an immediate consequence of Proposition 5.1 one obtains

**Proposition 5.2** In the local coordinates of Proposition 5.1 one finds that
\[ Q = \{(x,y,z,t) \in \Omega; \ q(x,y,z,t) = 0\}, \] (5.8)
where
\[ q = q_0 + q', \ q_0 = y^2 - 4xz, \ q' \in M_4^{4-3-2}(\Omega). \] (5.9)
See [17] for a proof. Now we deduce from it more information about the interaction of $Q$ and $\Sigma$.

**Proposition 5.3** With $P$ and $\Sigma$ as in Proposition 5.1, in a small neighborhood of $O$, there are smooth functions $F_i(z,t), \ 1 \leq i \leq 3$, such that $Q \cap \Sigma = B \cup E$
\[ B = \{z = z^3 F_1(z,t), \ y = z^2 F_2(z,t)\}, \] (5.10)
\[ E = \{z = \frac{3}{16} z^2 + z^3 F_3(z,t), \ y^2 = -\frac{27}{32} z^3 + z^4 F_4(z,t)\}. \] (5.11)
Away from $O$, $Q$ and $\Sigma$ meet transversally at $E$ and are tangent of third order at $B$. 

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6 Geometric Resolution

The hypersurfaces $\Sigma$ and $Q$ will be resolved to normal crossing by iterated quasi-homogeneous blow ups. As a first step we define the 4-3-2 blow up of $\mathbb{R}^n$ along $O = (0,0,0)$.

In $\mathbb{R}^3$ consider the non-round sphere

$$S^2_{4-3-2} = \{(\omega_1,\omega_2,\omega_3); \omega_1^6 + \omega_2^6 + \omega_3^{12} = 1\}$$

and the map

$$\beta_1 : X_1 = [0,\infty) \times S^2_{4-3-2} \longrightarrow \mathbb{R}^3, \quad \beta_1(s,\omega) = (s^4\omega_1, s^3\omega_2, s^2\omega_3).$$

This is surjective and restricts to a diffeomorphism of $X_1 \setminus \partial X_1$ onto $\mathbb{R}^n \setminus K$. Moreover the $\mathbb{R}^+$ action (5.1) lifts to the standard multiplicative action on the factor $[0,\infty)$.

From these observations above it follows that the lifts of the hypersurfaces and the bicharacteristic $B$ in the model case are:

$$\Sigma^{(1)} = \text{clos}[\beta_1^{-1}(\Sigma \setminus O)] = \{16\omega_1\omega_2^4 - 4\omega_2^2\omega_3^3 - 128\omega_1^2\omega_3^2 + 144\omega_1\omega_2\omega_3^2 + 256\omega_3^4 - 27\omega_2^2 = 0\},$$

$$Q_0^{(1)} = \text{clos}[\beta_1^{-1}(Q_0 \setminus O)] = \{\omega_2^2 - 4\omega_1\omega_3 = 0\}, \quad \text{(6.2)}$$

$$B_0^{(1)} = \text{clos}[\beta_1^{-1}(B \setminus O)] = \{\omega_1 = 0, \omega_2 = 0\}. \quad \text{(6.3)}$$

Fig 4: 
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$\Sigma^{(1)}$ has a cusp singularity at

$$L^{(1)} = \text{clos}[\beta^{-1}(L \setminus O)] = \{\omega_1 = -\frac{1}{12} \omega_2^2, \omega_2 = \frac{2}{3} \omega_3^3\}$$  \hspace{1cm} (6.4)$$

and a self-intersection at

$$H^{(1)} = \text{clos}[\beta^{-1}(L \setminus O)] = \{\omega_1 = \frac{1}{4} \omega_2^2, \omega_2 = 0\}. \hspace{1cm} (6.5)$$

For reasons that will become clear later on, there are two "great circles" on $S^2_{3-2}$ that will have to be taken into consideration. We define

$$C_1 = \{\omega_1 = 0, \ r = 0\}, \hspace{1cm} (6.6)$$

$$C_2 = \{\omega_3 = 0, \ r = 0\}. \hspace{1cm} (6.7)$$

More generally we find, see [17]

**Proposition 6.1** In local coordinates in which (3.1) and (5.8) hold the lifts $\Sigma^{(1)}, Q^{(1)}$ and $B^{(1)}$ of the hypersurfaces and the bicharacteristic to $X_1$ are diffeomorphic, on $X_1$, to the model $\Sigma^{(1)}, Q_0^{(1)}$ and $B^{(1)}$ under a diffeomorphism fixing $\partial X_1$ pointwise. Conversely any diffeomorphism preserving (3.1), (5.8) and $O$, lifts to a diffeomorphism of $X_1$ near $\partial X_1$ preserving $\Sigma^{(1)}$ and $Q^{(1)}$.

The full resolution of the geometry is obtained by blow ups of the three (really six) submanifolds $L^{(1)}, D_0^{(1)} = Q^{(1)} \cap C_2$ and $B^{(1)}$. There are local coordinates $(s,X,Y,T)$ near $L^{(1)}$ with

$$\Sigma^{(1)} = \{Y^3 = X^2\}, \hspace{1cm} (6.8)$$

near $D_0^{(1)}$ with

$$Q^{(1)} = \{X = Y^2\}, C_2 = \{X = 0, r = 0\}. \hspace{1cm} (6.9)$$

near $B^{(1)}$ with

$$Q^{(2)} = \{X = 0\}, \ \Sigma^{(1)} = \{X = Y^4\}, \ C_1 = \{X = Y^2, r = 0\}. \hspace{1cm} (6.10)$$

Thus $\Sigma^{(1)}$ can be resolved to normal crossing by a $3 - 2$ blow-up of $L^{(1)}$, thus set

$$S^3_{3-2} = \{((\theta_1, \theta_2) \in \mathbb{R}^2; \theta_1^4 + \theta_2^6 = 1\} \hspace{1cm} (6.11)$$

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and in local coordinates (6.8) we construct the map

\[ \beta_{3-2} : [0, \infty) \times [0, \infty) \times S^1_{3-2} \times \mathbb{R}^{n-3} \rightarrow X_1 \]

\[ \beta_{3-2}(s, r, \theta) = (r, s^3 \theta_1, s^2 \theta_2). \]  

(6.12)  
(6.13)

Fig 5:

It will also be necessary to blow-up \( D^{(1)}_0 \) with homogeneity 2-1-1, thus let

\[ S^2_{2-1-1} = \{ (\theta_1, \theta_2, \theta_3) \in \mathbb{R}^2; \theta_1^2 + \theta_2^4 + \theta_3^4 = 1 \} \]  
(6.14)

and in local coordinates (6.9) construct the map

\[ \beta_{2-1-1} : [0, \infty) \times [0, \infty)_R \times S^1_{2-1} \times \mathbb{R}^{n-3} \rightarrow X_1 \]

\[ \beta_{2-1}(s, R, \omega, t) = (R, s^2 \theta_1, s \theta_2, s \theta_3, t). \]  

(6.15)  
(6.16)

Fig 6:

To resolve \( Q^{(1)} \), \( \Sigma^{(1)} \) and \( C_1 \) to normal crossing it will be more convenient to use four normal blow-ups as in [12]. Since \( Q^{(1)} \) and \( \Sigma^{(1)} \) are tangent to third order at \( B^{(1)} \), if \( C_1 \) did not have to be taken into consideration, one could use a 4-1 nonhomogeneous blow-up to resolve \( Q^{(1)} \) and \( \Sigma^{(1)} \) to normal
crossing, but $C_1$ destroys the 4-1 homogeneity.

Fig 7:

Since $D^{(1)}, L^{(1)}$ and $B^{(1)}$ are disjoint we can use these maps to replace small neighborhoods of $D^{(1)}, L^{(1)}, B^{(1)}$ by their respective blow ups and so define the manifold with corners $X$ and a blow down map $\beta : X \rightarrow X_1$. Let

$$\beta = \beta_2 \circ \beta_1 : X \rightarrow \mathbb{R}^n$$

(6.17)

Denote

$$Q^{(2)} = \text{clos}[\beta_2^{-1}(Q^{(1)} \setminus (B^{(1)} \cup D_0^{(1)}))],$$

$$\Sigma^{(2)} = \text{clos}[\beta_2^{-1}(\Sigma^{(1)} \setminus (L^{(1)} \cup B^{(1)}))]$$

$$L^{(2)} = \text{clos}[\beta_2^{-1}(L^{(1)})],$$

$$B^{(2)} = \text{clos}[\beta_2^{-1}(B^{(1)})],$$

$$C_1^{(2)} = \text{clos}[\beta_2^{-1}(C_1 \setminus B^{(1)})],$$

$$C_2^{(2)} = \text{clos}[C_2 \setminus D_0^{(1)}].$$

The circle $C_2^{(2)}$ does not continue into the boundary face introduced by the 2-1-1 blow-up.

The manifold with corners $X$ has twelve boundary hypersurfaces which meet transversally pairs or triples. Let $\rho_L, \rho_D, 1 \leq j \leq 8, \rho_D$ and $\rho_K$ be respectively the defining functions of $\beta^{-1}(L)$, each of the eight hypersurfaces of $\beta^{-1}(B), \beta^{-1}(D)$ and $\beta^{-1}(K)$ (These functions are assumed to be extended smoothly past the surfaces they define).
Proposition 6.2 Under the $C^\infty$ map $\beta : X \to \mathbb{R}^n$ the lifts

$$\beta^*(M) = \text{clo} \beta^{-1}(M \setminus [K \cup L \cup B]),$$

(6.18)

for $M = Q, \Sigma$ are smooth hypersurfaces that intersect the boundaries of $X$ transversally. Any $C^\infty$ diffeomorphism of $X_1$ preserving $\Sigma^{(1)}, Q^{(1)} D_0^{(1)}$ and $\partial X_1$ lifts to a $C^\infty$ diffeomorphism of $X$ preserving all boundaries and all the hypersurfaces.

Let $L^2_c(X)$ be the space of compactly supported square integrable functions in $X$ with respect to the measure $\mu = \beta^*(dxdydz)$. Then the blow down map $\beta$ gives an isomorphism

$$\beta^* : L_c(\mathbb{R}^n) \leftrightarrow L^2_c(X).$$

(6.19)

Let $\mathcal{W}$ be the Lie algebra and smooth vector fields $W$ on $X$ satisfying the following properties:

1) $W$ is tangent to all boundary hypersurfaces.
2) $W$ is tangent to $\beta^*(\Sigma)$ and to $\beta^*(Q)$.
3) $W$ is tangent to $\mathcal{C}^{(2)}_1$.
4) In local coordinates $(r, s, X)$ in which $p_\Sigma = r$ and $C_1^{(2)} = \{r = X = 0\}$, $\mathcal{W}$ is spanned by $r\partial_r, s\partial_s, X\partial_X, r^2\partial_X$.

We then define for any integer $k$

$$J_k(\Omega) = \{u \in L^2_c(\Omega) : \beta^* u \in I_k L^2_c(X, \mathcal{W})\}$$

(6.20)

As a consequence of Propositions 6.1 and 6.2 it follows that the spaces $J_k(\Omega)$ are independent on the choices of coordinates. Moreover from the Gagliardo-Nirenberg type inequalities of [15] one obtains

Proposition 6.3 For any $k \in \mathbb{N}$, $J_k(\Omega) \cap L^\infty_{loc}(\Omega)$ is a $C^\infty$ algebra, i.e for any $f \in C^\infty(\mathbb{R}^m)$ and $u_i \in J_k(\Omega) \cap L^\infty(\Omega), 1 \leq i \leq m$,

$$f(u_1, \ldots, u_m) \in J_k(\Omega) \cap L^\infty_{loc}(\Omega).$$

(6.21)

By writing the generators of $\mathcal{V}(\Sigma, Q)$ and their lift under the map $\beta$ it is not hard to see that

$$J_k(\Omega) \subset I_k L^2_{loc}(\Omega, \mathcal{V}(\Sigma, Q))$$

(6.22)
7 The Linear Propagation Theorem

In this section we sketch the proof that the spaces \( J_k(\Omega) \) satisfy

**Theorem 7.1** Let \( f \in J_k(\Omega), \ f = 0 \text{ in } \Omega^- \). Let \( u \in H^1_{loc}(\Omega), \ u = 0 \text{ in } \Omega^- \), satisfy
\[
P u = f.
\] (7.1)

Then \( u, Du \in J_k(\Omega) \).

**Lemma 7.1** Let \( \phi \in C^\infty_0(X_1), \ \phi = 1 \text{ in sufficiently small neighborhoods of } L^{(1)}, E^{(1)} \text{ and } H^{(1)}, \ \phi = 0 \text{ outside slightly bigger neighborhoods.} \) There exist \( v_1, Dv_1 \in J_k(\Omega) \) such that
\[
\beta^*_1(Pv_1) - \phi \beta^*_1 f \in I_kL^2_{loc}(X_1, \partial X_1)
\] (7.2)

The proof of Lemma 7.1 is based on the fact that the lift of the operator \( P \) by the map \( \beta_1 \) is of real principal type in the totally characteristic sense, see [10], in some directions near \( L^{(1)}, E^{(1)} \text{ and } H^{(1)} \). One can then use the calculus of totally characteristic Fourier Integral Operators of [10] to transform the operator, the characteristic surfaces and their intersections into model cases. Lemma 7.1 is then a consequence of the mapping properties of these operators.

**Lemma 7.2** Let \( g \in L^2_{loc}(\Omega) \) be such that
\[
\beta^* g \in I_kL^2_{loc}(X, \partial X_1).
\] (7.3)

Then there exists \( v_2, Dv_2 \in J_k(\Omega) \) such that \( Pv_2 = g \).

The proof of Lemma 7.2 is considerably simpler than the one of Lemma 7.1, it is based on a commutator argument.

7.1 Marked Lagrangian Distributions

Let \( \Lambda \subset T^*\Omega \) be a smooth conic closed Lagrangian and let \( S_2 \subset S_1 \subset \Lambda \) be conic smooth hypersurfaces. Denote
\[
\mathcal{M}(\Lambda)_1 = \{ A \in \Psi^1(\Omega) : a = \sigma^1(A) = 0 \text{ at } \Lambda, \ H_a \text{ tangent to } S_1 \text{ and to } S_2 \} \tag{7.4}
\]
\[
I_kL^2_{loc}(\Omega, \mathcal{M}(\Lambda)_1) = \{ u \in L^2_c(\Omega) : \mathcal{M}(\Lambda)^j_1 u \subset L^2_{loc}(\Omega), \ j \leq k \}. \tag{7.6}
\]
A detailed study of these distributions can be found in [8]. As mentioned in Section 2, the marked Lagrangian Distributions were first introduced by Melrose in [9] to study the cusp case.

Let \( \Lambda_{\Sigma} = \text{clos}N^*(\Sigma_\text{reg}) \), \( \Lambda_Q = \text{clos}N^*(Q \setminus O) \). It is well known that \( \Lambda_{\Sigma} \) and \( \Lambda_Q \) are smooth conic Lagrangian submanifolds of \( T^*\mathbb{R}^3 \). Let \( \Lambda_B = N^*B \) and let \( \Lambda_O = T^*_O \mathbb{R}^3 \), denote \( S_1 = \Lambda_{\Sigma} \cap \Lambda_B = \Lambda_Q \cap \Lambda_B = \Lambda_{\Sigma} \cap \Lambda_Q \) and \( S_2 = \Lambda_{\Sigma} \cap \Lambda_O \). Let \( S_3 = \Lambda_Q \cap \Lambda_Q \) and let \( I_kL^2_{\text{loc}}(\Omega, \mathcal{M}(\Lambda_0)_3) \) be the space of marked Lagrangian distributions to \( \Lambda_0 \) marked by \( S_3 \) and \( S_2 \).

In coordinates where (3.1) holds one obtains that \( \mathcal{M}(\Sigma)_1 \) is the \( \Psi^0(\Omega) \) span of

\[
V_1 = 4x\partial_x + 3y\partial_y + 2z\partial_z, \quad V_2 = (2xz - \frac{3}{4}y^2)\partial_x - \frac{1}{2}yz\partial_y + 4x\partial_z, \\
P_1 = x(\partial_y^2 - \partial_x\partial_z), \quad P_2 = y(\partial_y^2 - \partial_x\partial_z), \\
P_3 = 4\partial_x^2 + 2z\partial_z^2 + y\partial_y\partial_z, \quad P_4 = (\partial_y^2 - \partial_x\partial_z)\partial_z, \\
P_5 = (\partial_y^2 - \partial_x\partial_z)\partial_y.
\]

Times elliptic factors of the appropriate orders. The space of marked Lagrangian distributions to the swallowtail marked by \( S \) and \( S_1 \) is however too small for our purposes, we shall need a slightly bigger one. Let \( P'_5 = (3\partial_y^2 - 8\partial_x\partial_z - 12z\partial_z^2)\partial_y^2 \) and define the space of “supermarked” Lagrangian distributions to \( \Lambda_{\Sigma} \) \( S \) and \( S_1 \) as

\[
I_{3k}L^2_\epsilon(\Omega, \mathcal{M}(\Lambda_{\Sigma})_1)^s = \{ u \in L^2_\epsilon(\Omega) : V_1^{\alpha_1}V_2^{\alpha_2}P_1^{\alpha_3}P_2^{\alpha_4}P_3^{\alpha_5}P_4^{\alpha_6}P_5^{\alpha_7}u \in H^{-\ell}(\Omega), \quad \ell = \ell_1 + \ell_2 + \ell_3 + \ell_4 + 6\ell_5 \leq 3k \}.
\]

Where the superscript \( s \) is for “supermarked”. The spaces of supermarked Lagrangians was introduced by M. Zworski in [18] where a more detailed description of those spaces is given. One defines the space \( I_kL^2_\epsilon(\Omega, \mathcal{M}(\Sigma)_1)^s \) for all integers \( k \) by complex interpolation. One can easily show that

\[
I_kL^2_\epsilon(\Omega, \mathcal{M}(\Lambda_{\Sigma})_1)^s \subset I_kL^2_\epsilon(\Omega, \mathcal{M}(\Lambda_{\Sigma})_1)^s.
\]

Let

\[
M_k(\Omega) = I_kL^2_\epsilon(\Omega, \mathcal{M}(\Lambda_{\Sigma})_1)^s + I_kL^2_\epsilon(\Omega, \mathcal{M}(\Lambda_Q)_1) + I_kL^2_\epsilon(\Omega, \mathcal{M}(\Lambda_B)_1) + I_kL^2_\epsilon(\Omega, \mathcal{M}(\Lambda_O)_3)
\]

be the space of marked Lagrangian distributions to \( \Sigma, Q \) and \( B \).
Lemma 7.3 Let $g \in J_k(\Omega)$ be such that $\beta^*g$ is supported away from $E^{(1)}$, $H^{(1)}$ and $L^{(2)}$, then $g \in M_k(\Omega)$.

The proof of Lemma 7.3 is quite long and consists basically of lifting the generators of each of the components of $M_k$ under the map $\beta$. Now we are going to use the same idea as in the case of the cusp, first we prove a propagation theorem for $M_k(\Omega)$ and then use again the equation to show that the solution is in fact in the smaller space $J_k(\Omega)$. By commutator methods one can prove

Lemma 7.4 Let $f \in M_k(\Omega)$, there exist $v_3, Dv_3 \in M_k(\Omega)$ such that $Pv_3 = f$.

Then one proves an elliptic regularity type of Theorem which states that

Lemma 7.5 Let $v, Dv \in M_k(\Omega)$ be such that $Pv \in J_k(\Omega)$. Then $v, Dv \in J_k(\Omega)$.

When one lifts $v \in M_k(\Omega)$ under the map $\beta$ one finds that it may be singular at some circles at the boundary of $X$, but it turns out that the lift of operator $P$ under the map $\beta$ is elliptic in some directions of $bT^*X$ over those circles and therefore one concludes that if $v$ satisfies the inclusion $Pv \in J_k(\Omega)$, then $v \in J_k(\Omega)$. This is the reason why one has to include the great circles in the definition of the spaces, since the hypersurfaces $\{z = 0\}$ and $\{z = 0\}$ are characteristic for $P_0$ the lift of the operator $P$ could not be elliptic on circles $C_1^{(2)}$ and $C_2^{(2)}$.

Conclusion of the proof of Theorem 7.1:

Let $v_1, v_2$ and $v_3$ be as in Lemmas 7.1, 7.2 and 7.3 and $w = u - v_1 - v_2 - v_3$. Then

$$Pw = 0, \quad w \in J_k(\Omega) \text{ in } t < 0. \quad (7.14)$$

Let

$$\mathcal{M}(\Lambda_Q \cup \Lambda_\Sigma) = \{ A \in \Psi^1(\Omega) : a = \sigma^1(A) = 0 \text{ on } \Lambda_Q \cup \Lambda_\Sigma \} \quad (7.15)$$

Equation (7.14) implies that

$$w, Dw \in L^2_{\text{loc}}(\Omega^-, \mathcal{M}(\Lambda_Q \cup \Lambda_\Sigma)). \quad (7.16)$$

By commutator methods one can easily show that

$$w, Dw \in L^2_{\text{loc}}(\Omega, \mathcal{M}(\Lambda_Q \cup \Lambda_\Sigma)). \quad (7.17)$$
By the arguments used in the proof of Lemma 7.3 one can show that
\[ I_k L^2_{\text{loc}}(\Omega, M(\Lambda Q \cup \Lambda \Sigma)) \subset J_k(\Omega). \]  
(7.18)

This concludes the proof of Theorem 7.1.

References


**Department of Mathematics**  
**Purdue University**  
**West Lafayette, IN 47907.**