NILS DENCKER
Preparation theorems for systems

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1. INTRODUCTION

The Malgrange preparation theorem is a useful tool in analysis. It is a generalization of the Weierstrass' preparation theorem to $C^\infty$ functions as follows: if $f(t, x) \in C^\infty(\mathbb{R} \times \mathbb{R}^d)$ satisfies

\begin{equation}
0 = f(0, 0) = \partial_t f(0, 0) = \cdots = \partial_t^{n-1} f(0, 0) \quad \text{and} \quad \partial_t^n f(0, 0) \neq 0,
\end{equation}

then we can factor

\begin{equation}
f(t, x) = c(t, x)(t^n + a_{n-1}(x)t^{n-1} + \cdots + a_1(x)t + a_0(x))
\end{equation}

near (0,0), where $c(0, 0) \neq 0$ and $a_j(0) = 0$, $0 \leq j < n$. The condition (1.1) means that

$f(t, 0) = c(t)t^n$,

where $c(0) \neq 0$. A possible generalization of this result to matrix valued functions, is to replace (1.1) by

\begin{equation}
0 = F(0, 0) = \partial_t F(0, 0) = \cdots = \partial_t^{n-1} F(0, 0) \quad \text{and} \quad |\partial_t^n F(0, 0)| \neq 0,
\end{equation}

where $F(t, x) \in C^\infty$ is $N \times N$ matrix valued, and $|F|$ is the determinant. Then we should obtain (1.2) with matrix valued $c(t, x)$ and $a_j(x)$, satisfying $|c(t, x)| \neq 0$ and $a_j(0) = 0$, $\forall j$. In the case when $n = 1$ in (1.3), this was proved in [1]. But the condition (1.3) is too restrictive, since it does not cover the cases when $F(t, x) = (f_j(t, x)\delta_{jk})$ is diagonal, with diagonal elements $f_j$ satisfying (1.1) with different $n$ (in which case we can use the Malgrange preparation theorem). More generally, we assume that

\begin{equation}
F(t, 0) = C(t) \sum_{j=0}^n t^j \pi_j,
\end{equation}

where $|C(0)| \neq 0$, and $\pi_j$ is orthogonal projection on $\mathbb{C}^N$, such that $\pi_j \pi_k = \delta_{jk} \pi_k$ and $\sum_{j=0}^n \pi_j = \text{Id}_N$. This includes condition (1.3), and is equivalent to

\begin{equation}
\mathbb{C}^N = \bigoplus_{j=0}^n \text{Im} \partial_t^j F(0, 0) \bigg|_{E_{j-1}},
\end{equation}

where $E_k = \bigcap_{0 \leq j \leq k} \text{Ker} \partial_t^j F(0, 0)$. This condition is invariant under left multiplication of $F$ by elliptic systems. Assuming (1.5), we show in Theorem 2.5 that

\begin{equation}
F(t, x) = C(t, x) \left( \sum_{j=0}^n t^j \pi_j + \sum_{j=0}^{n-1} t^j A_j(x) \right),
\end{equation}

where

\begin{equation}
\sum_{j=0}^n t^j \pi_j + \sum_{j=0}^{n-1} t^j A_j(x) = C(t, x) \left( \sum_{j=0}^n t^j \pi_j + \sum_{j=0}^{n-1} t^j A_j(x) \right).
\end{equation}
near \((0,0)\), where \(C(t,x)\) and \(A_j(x)\) are \(C^\infty\) functions satisfying \(|C(0,0)| \neq 0\) and \(A_j(0) = 0, 0 \leq j < n\). Since we are allowed to make row operations, we obtain that \(A_j(x)\pi_k \equiv 0\) when \(j \geq k\). The orthogonal projections \(\pi_k\) are uniquely determined in (1.6).

Now condition (1.5) is also too restrictive, for example, it is not satisfied for the polynomials

\[
\sum_{j=0}^{n} t^j \pi_j + \sum_{j=0}^{n-1} t^j A_j(x),
\]

when \(A_j(x)\pi_k \equiv 0\) for \(j \geq k\), and \(A_j(0) \neq 0\) for some \(j\). But such systems always satisfy the condition

\[
(1.7) \quad \partial_t^m (\det F)(0,0) \neq 0,
\]

for some \(m\), thus the determinant does not vanish of infinite order. In Theorem 3.3, we show that condition (1.7) is sufficient for a preparation of \(F\) on the form (1.6), with orthogonal projections \(\{\pi_j\}\) on \(C^N\), such that \(\pi_j \pi_k = \delta_{jk} \pi_k\) and \(\sum_{j=0}^{n} \pi_j = \text{Id}_N\). We obtain that \(A_j(x)\) satisfies \(A_j(x)\pi_k \equiv 0\) when \(j \geq k\), and

\[
(1.8) \quad A_j(0) = \sum_{i<j<k} \pi_i A_j(0) \pi_k.
\]

The projections \(\pi_k\) and matrices \(A_j(0)\) are uniquely determined by (1.8). The rank of the projections \(\pi_k\) are determined by the elementary divisors of the Taylor expansion of \(F(t,0)\) at \(t = 0\), but the projections themselves are harder to compute, except for \(\pi_0\) and \(\pi_1\) (see Remark 3.4).

By allowing right multiplication by elliptic systems, i.e. column operations, we may also obtain that \(\pi_k A_j(x) \equiv 0\) when \(j \geq k\), and \(A_j(0) = 0\) when (1.8) holds (see Proposition 3.5). By duality, we obtain the corresponding results for right preparation of \(F\), i.e. left preparation of \(F^*\), in Theorem 4.2. We also prove the generalization of Malgrange's division theorem in Theorems 3.6 and 4.3. The method of proof follows in part Mather [6], with the improvements of Hörmander [2, Section 7.5]. Observe that, since the proofs of Malgrange [3] use commutative algebra, they are not directly applicable here.

2. LEFT PREPARATION

In what follows, let \(\pi_j\) be (complex) orthogonal projections in \(C^N\), \(0 \leq j \leq n\), such that \(\sum_{j=0}^{n} \pi_j = \text{Id}_N\) and \(\pi_j \pi_k = \delta_{jk} \pi_k\). This means that \(\pi_j^* = \pi_j, \forall j\). Put

\[
P(t,A) = \sum_{0 \leq j \leq n} t^j \pi_j + \sum_{0 \leq j < n} t^j A_j,
\]

with \(A = (A_0, \ldots, A_{n-1})\), where \(A_j \in \mathcal{L}_N = \mathcal{L}(C^N, C^N)\) is a complex \(N \times N\) matrix satisfying

\[
(2.2) \quad A_j \pi_k = 0 \quad \text{when} \quad j \geq k.
\]

Let \(|A_j| = \det A_j\) be the determinant of \(A_j\), and let \(||A|| = \sum_j ||A_j||\), where \(||A_j||\) is the matrix norm. We are going to divide matrix valued analytic functions with such matrix valued polynomials. Let \(\omega\) be an open set in \(C\), let \(G(t)\) be analytic in \(\overline{\omega}\) with values in \(\mathcal{L}_N\), and assume \(|P(t,A)| \neq 0\) on \(\partial \omega \in C^1\). Then

\[
(2.3) \quad G(t) = Q(t)P(t,A) + R(t) \quad t \in \omega,
\]
where

\[ Q(t) = (2\pi i)^{-1} \int_{\partial \omega} G(s)P(s, A)^{-1}(s - t)^{-1} \, ds \quad t \in \omega \]

is analytic in \( \omega \), and

\[ R(t) = (2\pi i)^{-1} \int_{\partial \omega} G(s)P(s, A)^{-1}(P(s, A) - P(t, A))(s - t)^{-1} \, ds \]

is a polynomial of degree \( n - 1 \) in \( t \). We find that

\[ R(t)\pi_k = (2\pi i)^{-1} \int_{\partial \omega} G(s)P(s, A)^{-1}((s^k - t^k)\pi_k + \sum_{j < k} (s^j - t^j)A_j \pi_k)(s - t)^{-1} \, ds \]

is a polynomial of degree \( < k \) in \( t \). The remainder \( R(t) \) is uniquely determined by this condition, if \( R(t)P(t, A)^{-1} \) is analytic when \( t \not\in \omega \).

Let \( V \subseteq \bigotimes_{j=0}^{n-1} \mathcal{L}_N \) be the set of \( A = (A_0, \ldots, A_{n-1}) \) satisfying (2.2), let \( m_k = \text{Rank} \pi_j \) and \( m = \sum_{1 \leq j \leq n} j \cdot m_j \). Since \( A_k = \sum_{k < j} A_k \pi_j \), \( A_k \) lies in a subspace of (complex) dimension \( \sum_{k < j} m_j N \) of \( \mathcal{L}_N \). This implies that \( V \cong C^m \cong R^{2m} \), since we have \( \sum_{0 \leq k < j \leq n} m_j = \sum_{k=1}^{n-1} j \cdot m_j = m \). We obtain the following division theorem on \( R \).

**Proposition 2.1.** Let \( F(t) \in \mathcal{S}(R) \) have values in \( \mathcal{L}_N \). Then we can find \( Q(t, A, F) \in C^\infty(R \times V) \) and \( R_j(A, F) \in C^\infty(V) \), \( 0 \leq j < n \), with values in \( \mathcal{L}_N \) and depending linearly on \( F(t) \), such that \( R_j(A)\pi_k \equiv 0 \) when \( j > k \), and

\[ F(t) = Q(t, A, F)P(t, A) + \sum_{j=0}^{n-1} t^j R_j(A, F) \quad \text{when} \quad \|A\| < 1, \quad t \in R. \]

We also get global estimates on all the derivatives of \( Q \) and \( R_j \). The proof of Proposition 2.1 follows the proof of [2, Lemma 7.5.4]. Thus, first we show that we may divide bounded analytic functions by \( P(t, A) \) in a strip containing \( R \), with uniform bounds. Then, we get the result by a Fourier decomposition of \( F \).

**Remark 2.2.** If \( F(t, x) \in \mathcal{S}(R \times R^n) \) depends on parameters \( x \), then \( Q(t, A, F(\cdot, x)) \in C^\infty(R \times V \times R^n) \) and \( R_j(A, F(\cdot, x)) \in C^\infty(V \times R^n) \). In fact, by linearity and continuity, we may differentiate directly on \( F \).

Next, we shall compute some invariants. Let \( F(t) \) be a \( C^\infty \) function on \( R \) with values in \( \mathcal{L}_N \). Put \( E_{-1} = C^N \) and

\[ E_k = \bigcap_{0 \leq j \leq k} \ker \partial_j^i F(0), \quad k \geq 0. \]

**Proposition 2.3.** If

\[ C^N = \bigoplus_{j=0}^{n} \text{Im} \partial_j^i F(0) \bigg|_{E_{j-1}}, \]

then it follows that \( E_n = \{ 0 \} \). We find that the spaces \( E_k, 0 \leq k \leq n \), and condition (2.9) are invariant under left multiplication of \( F \) by elliptic systems.

**Proof:** Assume that \( C(t) \) is elliptic, then \( \ker CF(0) = \ker F(0) \). Now, we have by Leibniz’ rule

\[ \partial^k CF(0) = \sum_{j=0}^{k} \binom{k}{j} \partial^k-j C(0) \partial^j F(0), \]

where
so by induction we obtain
\[ \bigcap_{0 \leq j \leq k} \ker \partial_j^i CF(0) = \left( \bigcap_{0 \leq j < k} \ker \partial_j^i F(0) \right) \cap \ker \partial_k^i (CF)(0) = \bigcap_{0 \leq j \leq k} \ker \partial_j^i F(0), \]
which gives the invariance of \( E_k \), \( \forall \ k \). We also obtain that
\[ (2.10) \quad \text{Im} \partial_i^k (CF)(0) \big|_{E_{k-1}} = C(0) \text{Im} \partial_i^k F(0) \big|_{E_{k-1}}. \]
Since \(|C(0)| \neq 0\), this gives the invariance of condition (2.9).
It remains to prove that \( \dim E_n = 0 \). Let \( m_k = \dim E_k \), so that \( m_{-1} = N \). Then, we find
\[ \dim \left( \text{Im} \partial_i^k F(0) \big|_{E_{k-1}} \right) = \dim E_{k-1} - \dim \left( \ker \partial_i^k F(0) \big|_{E_{k-1}} \right) = m_{k-1} - m_k. \]
Thus we find from (2.9) that
\[ N \leq \sum_{j=0}^{n} (m_{k-1} - m_k) = N - m_n. \]
This means that \( m_n \leq 0 \), which proves the result. \( \blacksquare \)

**Proposition 2.4.** Let \( C^N = E_{-1} \supseteq E_0 \supseteq \cdots \supseteq E_n = \{0\} \), and let \( \pi_k \) be the orthogonal projection on \( E_k \cap E_{k-1} \), for \( 0 \leq k \leq n \). Then it follows that \( \pi_j \pi_k = \delta_{jk} \pi_k \), and
\[ (2.11) \quad \bigoplus_{j=0}^{k} \text{Im} \pi_j = E_k^\perp, \quad 0 \leq k \leq n. \]
In particular, we obtain \( \bigoplus_{0 \leq j \leq n} \text{Im} \pi_j = C^N \), which implies \( \sum_{j=0}^{n} \pi_j = \text{Id}_N \).
**Proof:** Clearly, \( \ker \pi_k = (\text{Im} \pi_k)^\perp = E_k \oplus E_{k-1}^\perp \), so
\[ \text{Im} \pi_j \subseteq E_{j-1} \subseteq E_k \subseteq \ker \pi_k \quad \text{if} \quad j > k. \]
Thus, \( \pi_k \pi_j \equiv 0 \) when \( j > k \), and by taking adjoints we obtain this when \( j < k \), which implies \( \pi_j \pi_k = \delta_{jk} \pi_k \).
By taking orthogonal complements, we find that (2.11) is equivalent to
\[ (2.12) \quad \bigcap_{j=0}^{k} \ker \pi_j = E_k \quad 0 \leq k \leq n. \]
We find \( \ker \pi_0 = E_0 \oplus E_{-1}^\perp = E_0 \). Assume by induction that (2.12) holds for some \( k \geq 0 \). Then we find that
\[ \bigcap_{0 \leq j \leq k+1} \ker \pi_j = E_k \cap (E_{k+1} \oplus E_{k+1}^\perp) = E_{k+1}, \]
since \( E_{k+1} \subseteq E_k \), thus by induction we obtain (2.12) for all \( k \). Since \( E_n = \{0\} \) we find that \( \sum_{j=0}^{n} \pi_j \) is bijective, and since \( (\sum_{j=0}^{n} \pi_j)^2 = \sum_{j=0}^{n} \pi_j \), it is equal to the identity. \( \blacksquare \)
Now, we can state the following generalization of the Malgrange preparation theorem.
Theorem 2.5. Let $F(t, x)$ be a $C^\infty$ function of $(t, x)$ in a neighborhood of the origin of $\mathbf{R} \times \mathbf{R}^d$, with values in $\mathcal{L}_N$, and assume that

\begin{equation}
C^N = \bigoplus_{j=0}^{n} \text{Im} \partial_t^j F(0, 0) \bigg|_{E_{j-1}},
\end{equation}

where $E_{-1} = C^N$ and $E_k = \bigcap_{0 \leq j \leq k} \text{Ker} \partial_t^j F(0, 0)$. Let $\pi_k$ be the orthogonal projection on $E_k \cap E_{k-1}$ for $0 \leq k \leq n$. Then, we may factor

\begin{equation}
F(t, x) = C(t, x) \left( \sum_{j=0}^{n} \partial_t^j \pi_j + \sum_{j=0}^{n-1} \partial_t^j A_j(x) \right) = C(t, x) P(t, A(x))
\end{equation}

near $(0, 0)$, where $C(t, x)$ and $A_j(x)$ are $C^\infty$ functions with values in $\mathcal{L}_N$, satisfying $A_j(x) \pi_k \equiv 0$, $j \geq k$. We also find $|C(0, 0)| \neq 0$ and $A_j(0) = 0$, $0 \leq j < n$. If $F$ is real (matrix) valued, we may choose $C$ and $A_j$ real (matrix) valued (and the projections $\pi_k$ are real).

This is proved by using Proposition 2.1 and Remark 2.2, to divide $F(t, x)$ by the polynomials $P(t, A)$. Then, we use the implicit function theorem, to choose $A(x) \in C^\infty$ making the remainder $\sum_j t^j R_j(A(x), F) \equiv 0$.

Proposition 2.3 and (2.13) imply that $E_n = \{ 0 \}$. Thus, we obtain from Proposition 2.4 that $\pi_j \pi_k = \delta_{jk} \pi_k$ and $\sum_j \pi_j = \text{Id}_N$. Since (2.14) implies

\[ F(t, 0) = C(t, 0) \sum_{j=0}^{n} t^j \pi_j, \]

we obtain from Proposition 2.3 that the condition (2.13) is necessary for the preparation (2.14).

Example 2.6. Let $F(t, x)$ be a $C^\infty$ function of $(t, x)$ with values in $\mathcal{L}_N$, and assume that

\[ |\partial_t^j F(0, 0)| \neq 0 \quad \text{and} \quad \partial_t^j F(0, 0) \equiv 0, \quad 0 \leq j < n. \]

Then we obtain from Theorem 2.5

\[ F(t, x) = C(t, x) \left( t^n \text{Id}_N + \sum_{0 \leq j < n} t^j A_j(x) \right), \]

where $C(t, x)$ and $A_j(x)$ are $C^\infty$ functions with values in $\mathcal{L}_N$, $|C(0, 0)| \neq 0$ and $A_j(0) = 0$, $0 \leq j < n$. (The case when $n = 1$ was proved in [1, Theorem A.3].)

3. The Preparation Theorem

The condition (2.13) in Theorem 2.5 is still too restrictive. In fact, the systems $P(t, A(x))$ in (2.14) do not satisfy condition (2.13) when $A(0) \neq 0$, but will be acceptable normal forms when $A(x) \in \mathbf{V}$, i.e. $A_j(x) \pi_k \equiv 0$ for $j \geq k$. As before, we assume that $\pi_j$ is orthogonal projection in $C^N$, $0 \leq j \leq n$, such that $\sum_{j=0}^{n} \pi_j = \text{Id}_N$ and $\pi_i \pi_j = \delta_{ij} \pi_j$. First, we consider the necessary condition for such a preparation.
PROPOSITION 3.1. Let \( F(t) \in C^\infty(\mathbb{R}) \) with values in \( \mathcal{L}_N \), and assume that

\[
F(t) = C(t) \left( \sum_{j=0}^{n} t^j \pi_j + \sum_{j=0}^{n-1} t^j A_j \right) = C(t)P(t, A)
\]

where \(|C(0)| \neq 0\) and \( A_j \pi_k = 0 \) when \( j \geq k \). Then it follows that

\[
\partial_t^m (\det F)(0) \neq 0,
\]

for some \( m \). We also find

\[
E_n = \bigcap_{0 \leq k \leq n} \Ker \partial_t^k F(0) = \{0\}.
\]

PROOF: Since the spaces \( E_k \) are invariant under multiplication from left by elliptic systems by Proposition 2.3, we may replace \( F(t) \) by \( P(t, A) \) in (3.3). Now \( \partial_t^k P(0, A) = k!(\pi_k + A_k) \), where \( A_k = \sum_{k<j} A_k \pi_j \). Thus, we find that \( \Ker \partial_t^n F(0) = \Ker \pi_n \). By induction we have

\[
\bigcap_{j=k}^{n} \Ker(\pi_j + A_j) = \left( \bigcap_{j=k+1}^{n} \Ker \pi_j \right) \cap \Ker(\pi_k + A_k) = \bigcap_{j=k}^{n} \Ker \pi_j,
\]

for \( 0 \leq k \leq n \), which proves (3.3). It is also clear that condition (3.2) is invariant under multiplication by elliptic systems. By a (constant) orthogonal base change, we may assume that

\[
\text{Im} \pi_k = \{(z_1, \ldots, z_N) : z_j \neq 0 \Rightarrow \sum_{i=0}^{k-1} \text{Rank} \pi_i < j \leq \sum_{i=0}^{k} \text{Rank} \pi_i \}, \quad 0 \leq k \leq n.
\]

Since \( P(t, A)\pi_k = (t^k + \sum_{j<k} t^j A_j)\pi_k \) we find that

\[
\partial_t^m (\det P)(0, A) = \left| \sum_{k=0}^{n} k!\pi_k \right| \neq 0,
\]

if \( m = \sum_{j=1}^{n} j \cdot \text{Rank} \pi_j \), which proves (3.2). ∎

The factorization (3.1) is not unique, according to the following

EXAMPLE 3.2. Let

\[
\begin{align*}
P_1(t) &= \begin{pmatrix} t & 1 \\ 0 & t \end{pmatrix} = t\text{Id}_2 + A_0 \\
P_2(t) &= \begin{pmatrix} t^2 & 0 \\ t & 1 \end{pmatrix} = \pi_0 + t^2 \pi_2 + tB_1
\end{align*}
\]

and \( Q(t) = \begin{pmatrix} t & 1 \\ 1 & 0 \end{pmatrix} \). Then we have \( Q(t)P_1(t) = P_2(t) \), and \( |Q(t)| \equiv 1 \). Since \( B_1 \pi_0 = 0 \), it is clear that \( P_1(t) \) and \( P_2(t) \) are on the form (2.1)–(2.2).

Now we are ready to state the main preparation theorem.
Theorem 3.3. Let \( F(t, x) \) be a \( C^\infty \) function of \((t, x)\) in a neighborhood of the origin of \( \mathbb{R} \times \mathbb{R}^d \) with values in \( L_N \), and assume that
\[
\partial_t^m (\det F)(0, 0) \neq 0 \quad \text{and} \quad \partial_t^k (\det F)(0, 0) = 0, \quad 0 \leq k < m.
\]
Then we may factor
\[
F(t, x) = C(t, x) \left( \sum_{j=0}^{n} t^j \pi_j + \sum_{j=0}^{n-1} t^j A_j(x) \right) = C(t, x)P(t, A(x))
\]
near \((0, 0)\), where \( \pi_j \) is orthogonal projection in \( \mathbb{C}^N \), \( 0 \leq j \leq n \), such that \( \pi_j \pi_k = \delta_{jk} \pi_k \) and \( \sum_{j=0}^{n} \pi_j = \text{Id}_N \). Here \( C(t, x) \) and \( A_j(x) \) are \( C^\infty \) functions with values in \( L_N \), satisfying \( |C(0, 0)| \neq 0 \), \( A_j(x) \pi_k \equiv 0 \) when \( j \geq k \) and
\[
A_j(0) = \sum_{i < j < k} \pi_i A_j(0) \pi_k,
\]
which implies \( A_0(0) = 0 \). The projections \( \pi_k \) and matrices \( A_j(0) \) are uniquely determined by the condition (3.6), and it follows that \( m = \sum_j j \cdot \text{Rank} \pi_j \) in (3.4). If \( F \) is real (matrix) valued, we may choose \( C(t, x), \pi_k \) and \( A_j(x) \) real (matrix) valued.

Theorem 3.3 is proved by reducing to the case of Theorem 2.5. Since condition (2.13) is necessary for that preparation, and is invariant under left multiplications, we must also multiply \( F \) from the right. Then, we have to be careful not to destroy the normal form \( P(t, A(x)) \).

Remark 3.4. The rank of the projections \( \pi_k \) are determined by the elementary divisors of the Taylor expansion of \( F(t, 0) \) at \( t = 0 \). In fact, let \( d_k \) be the determinant factors for \( 1 \leq k \leq N \), i.e. the greatest common divisor of the minors of order \( k \) of the Taylor expansion. Then \( \epsilon_k = d_k/d_{k-1} \) are the elementary divisors, and \( \text{Rank} \pi_j \) is the number of \( k \) such that \( \epsilon_k \) is divisible by \( t^j \) but not by \( t^{j+1} \) (see [8, § 85]). The projections \( \pi_j \) are harder to compute, except for \( j = 1, 2 \), since in these cases \( \text{Ker} \pi_0 = \text{Ker} F(0, 0), \) and
\[
\text{Ker} \pi_0 \cap \text{Ker} \pi_1 = \text{Ker} \left[ \partial_t F(0, 0) \right]_{\text{Ker} F(0, 0)} \hookrightarrow \mathbb{C}^N / \text{Im} F(0, 0) = \text{Coker} F(0, 0).
\]

By multiplication from right with invertible matrices, i.e. column operations, we may also obtain that \( \pi_k B_j(x) \equiv 0 \) when \( j \geq k \), and \( B(0) = 0 \) in (3.5), according to the following Proposition 3.5. Assume that \( \pi_j \) are orthogonal projections in \( \mathbb{C}^N \), \( 0 \leq j \leq n \), such that \( \pi_j \pi_k = \delta_{jk} \pi_k \) and \( \sum_{j=0}^{n} \pi_j = \text{Id}_N \). Let
\[
P(t, A(x)) = \sum_{j=0}^{n} t^j \pi_j + \sum_{j=0}^{n-1} t^j A_j(x)
\]
where \( A_j(x) \) are \( C^\infty \) functions with values in \( L_N \), satisfying \( A_j(x) \pi_k \equiv 0 \) for \( j \geq k \). Then we may find \( C(t, x) \in C^\infty \) with values in \( L_N \), such that \( |C(t, x)| \neq 0 \) and
\[
P(t, A(x))C(t, x) = P(t, B(x)),
\]
where \( \pi_k B_j(x) \equiv B_j(x) \pi_k \equiv 0 \) when \( j \geq k \). At the points \( x_0 \) where \( A_j(x_0) \) satisfies (3.6), \( \forall j \), i.e. \( \pi_k A_j(x_0) = 0 \) when \( k \geq j \), we obtain that \( B_k(x_0) = 0, \forall k \). When \( P(t, A) \) is real (matrix) valued, we may take \( C(t, x) \) and \( B_j(x) \) real (matrix) valued.

We also obtain the following generalization of the division theorem.
THEOREM 3.6. Let $F(t,x)$ satisfy the hypothesis in Theorem 3.3. If $G(t,x)$ is a $C^\infty$ function in a neighborhood of $(0,0)$ with values in $\mathcal{L}_N$, then we can write

\[ G(t,x) = Q(t,x)F(t,x) + \sum_{j=0}^{n-1} t^j R_j(x) \]

near $(0,0)$. Here $Q(t,x)$ and $R_j(x)$ are $C^\infty$ functions with values in $\mathcal{L}_N$, satisfying $R_j(x)\pi_k \equiv 0$ when $j \geq k$, for the projections $\pi_k$ in Theorem 3.3. If condition (2.13) also is satisfied, then $\pi_k$ is the orthogonal projection on $E_k^+ \cap E_{k-1}$ for $0 \leq k \leq n$, where $E_{-1} = \mathbb{C}^N$ and $E_k = \bigcap_{0 \leq j \leq k} \text{Ker } \partial_t^j F(0,0)$, $k \geq 0$.

PROOF: By Theorem 3.3, we may assume that

\[ F(t,x) = \sum_{j=0}^n t^j \pi_j + \sum_{j=0}^{n-1} t^j A_j(x) = P(t,A(x)), \]

where $A_j(x)\pi_k \equiv 0$ when $j \geq k$. Since it is no restriction to assume $G(t,x) \in C_0^\infty$, the statement follows from Proposition 2.1 and Remark 2.2, with $Q(t,x) = Q(t,A(x),G(\cdot,x))$ and $R_j(x) = R_j(A(x),G(\cdot,x))$ in (2.7). When $F(t,x)$ satisfies (2.13) also, we find that $\pi_k$ is the orthogonal projection on $E_k^+ \cap E_{k-1}$.

4. RIGHT PREPARATION

In Theorems 2.5 and 3.3, we have only done left preparation of matrix valued functions. By taking transposes we also obtain the corresponding results for right preparation. We first examine what condition we get on $F*$, when (2.13) holds for $F^*$. Let $F(t)$ be a $C^\infty$ function on $\mathbb{R}$ with values in $\mathcal{L}_N$, put $E_{-1}^* = \mathbb{C}^N$, and

\[ E_k^* = \bigcap_{0 \leq j \leq k} \text{Ker } \partial_t^j F^*(0), \quad k \geq 0. \]

Let $F_k$ be the mapping

\[ F_k : \mathbb{C}^N \ni w \longmapsto \partial_t^k F(0)w \quad (\text{mod } I_{k-1}) \quad k \geq 0, \]

where $I_{-1} = \{0\}$, and

\[ I_k = \bigoplus_{0 \leq j \leq k} \text{Im } \partial_t^j F(0), \quad k \geq 0. \]

PROPOSITION 4.1. The condition

\[ \mathbb{C}^N = \bigoplus_{k=0}^n \text{Im } \partial_t^k F^*(0) \big|_{E_{k-1}^*} \]

is equivalent to

\[ \{0\} = \bigcap_{0 \leq k \leq n} \text{Ker } F_k, \]

and implies

\[ \mathbb{C}^N = \bigoplus_{0 \leq k \leq n} \text{Im } \partial_t^k F(0). \]
We find that condition (4.4) and the spaces $I_k = \bigoplus_{0 \leq j \leq k} \text{Im} \partial_t^j F(0)$, $0 \leq k \leq n$, are invariant under right multiplication of $F$ by elliptic systems.

**Proof:** We have by duality that

\[
I_k = \bigoplus_{0 \leq j \leq k} \text{Im} \partial_t^j F(0) = \left( \bigcap_{0 \leq j \leq k} \text{Ker} \partial_t^j F^*(0) \right)^\perp = (E_k^*)^\perp.
\]

Let $\pi_k$ be the orthogonal projection on $I_k \cap I_{k-1}^\perp = (E_k^*)^\perp \cap E_{k-1}^*$, then we find $\text{Ker} F_k = \text{Ker} \pi_k \partial_t^k F(0)$ and

\[
\text{Im} \partial_t^k F^*(0) \bigg|_{E_{k-1}^*} = \text{Im} \partial_t^k F^*(0) \pi_k = (\text{Ker} \pi_k \partial_t^k F(0))^\perp.
\]

By Proposition 2.3, condition (4.3) is invariant under multiplication of $F$ by invertible systems from right, and it is equivalent to (4.4) by (4.7). We also obtain from Proposition 2.3 that the spaces $E_k^* = I_k^\perp$ are invariant under right multiplication of $F$ by invertible systems. Since condition (4.3) implies $E_n^* = \{0\}$ by Proposition 2.3, we obtain (4.5). $lacksquare$

Now we obtain from Theorems 2.5 and 3.3 the following result.

**Theorem 4.2.** Let $F(t,x)$ be a $C^\infty$ function of $(t,x)$ in a neighborhood of the origin of $\mathbb{R} \times \mathbb{R}^d$ with values in $\mathcal{L}_N$ satisfying (3.4). Then we may factor

\[
F(t,x) = \left( \sum_{j=0}^{n} t^j \pi_j + \sum_{j=0}^{n-1} t^j A_j(x) \right) C(t,x) = P(t,A(x))C(t,x)
\]

equation (4.8)

near $(0,0)$, where $\pi_j$ is orthogonal projection in $\mathbb{C}^N$, $0 \leq j \leq n$, such that $\pi_j \pi_k = \delta_{jk} \pi_k$ and $\sum_{j=0}^{n} \pi_j = \text{Id}_N$. Here $C(t,x)$ and $A_j(x)$ are $C^\infty$ functions with values in $\mathcal{L}_N$, satisfying $|C(0,0)| \neq 0$, $\pi_k A_j(x) \equiv 0$ when $j \geq k$, and

\[
A_j(0) = \sum_{i \geq j > k} \pi_i A_j(0) \pi_k,
\]

which implies $A_0(0) = 0$. The projections $\pi_k$ and matrices $A_j(0)$ are uniquely determined by condition (4.9), and $m = \sum_j j \cdot \text{Rank} \pi_j$ in (3.4). If also condition (4.4) is satisfied, we find that $A_j(0) = 0$, $0 \leq j < n$, and $\pi_k$ is the orthogonal projection on $I_k \cap I_{k-1}^\perp$ for $0 \leq k \leq n$, where $I_{-1} = \{0\}$, $I_k = \bigoplus_{0 \leq j \leq k} \text{Im} \partial_t^j F(0,0)$. If $F$ is real (matrix) valued, we may choose $C$, $\pi_k$ and $A_j$ real (matrix) valued.

It is clear that condition (3.4) is necessary for the preparation (4.8), and condition (4.4) is necessary when $A(0) \equiv 0$. We also obtain the following version of the division theorem from Theorem 3.6 by duality.

**Theorem 4.3.** Let $F(t,x)$ satisfy the hypothesis in Theorem 4.2. If $G(t,x)$ is a $C^\infty$ function in a neighborhood of $(0,0)$ with values in $\mathcal{L}_N$, then we can write

\[
G(t,x) = F(t,x)Q(t,x) + \sum_{j=0}^{n-1} t^j R_j(x)
\]

near $(0,0)$. Here $Q(t,x)$ and $R_j(x)$ are $C^\infty$ functions with values in $\mathcal{L}_N$, satisfying $\pi_k R_j(x) \equiv 0$ when $j \geq k$, for the projections $\pi_k$ in Theorem 4.2.
REFERENCES


Box 118, S-221 00 Lund, Sweden