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On resonant scattering for time-periodic perturbations

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1. The energy of a quantum system described by a time-dependent Hamiltonian $H(t)$ is not conserved. However, if a dependence of $H(t)$ on $t$ is periodic, it can be changed only by some integer number. In other words, the quasi-energy, i.e. the energy defined up to an integer, is a conserved quantity.

Here we discuss scattering of a plane wave by a time-periodic potential. Due to the quasi-energy conservation such a process is described by a set of amplitudes $S_n(\lambda)$ where $\lambda$ is energy of an incident wave (in other terms, of a quantum particle) and $n$ is arbitrary integer. We always decompose $\lambda$ as $\lambda = m + \theta$ where $m \in \mathbb{Z}$ is the entire part of $\lambda$ and $\theta \in [0,1]$. Each $S_n(\lambda)$ corresponds to a channel when energy is changed by $n - m$. Actually, amplitudes $S_n(\lambda)$ for $n \geq 0$ correspond to outgoing waves and amplitudes $S_n(\lambda)$ for $n < 0$ correspond to exponentially decaying modes. In some sense these modes play the role of bound or quasi-bound states for time-independent Hamiltonians. It means that they represent states which can have long though finite time of life. Thus exponentially decaying modes are essential for a detailed picture of interaction of an incident wave with a quantum system but they do not contribute to the scattering matrix of this process. Our aim is to study the transformation of exponentially decaying modes into proper bound states as a time-periodic perturbation is switched off.

In fact, we shall consider the following situation. Suppose that $H(t) = H_1 + \epsilon V(t)$ where the Hamiltonian $H_1$ has a negative eigenvalue $\lambda_1$ and the coupling constant $\epsilon$ is small. Physically, it is natural to conjecture that the bound state of the system with the Hamiltonian $H_1$ will give rise to some kind of long-living state
for the family $H(t)$. Due to the quasi-energy conservation this state is insignificant if energy $\lambda$ of an incident particle and $\lambda_1$ do not coincide by modulus of $\mathbb{Z}$. However, if energy $\lambda$ is resonant, that is $\lambda - \lambda_1 = K \in \mathbb{Z}$, then an incident particle can strongly interact with this quasi-bound state. Therefore the corresponding amplitude $S_{m-k}(\lambda, \varepsilon)$ is expected to be very large for small $\varepsilon$. Below we will show at the example of zero-range potentials that this physical picture is correct.

The problem of resonances for time-periodic perturbations was studied earlier by K. Yajima [1] in a different, more mathematical, framework. Our approach is closer to physical papers [2]–[5]. In particular, in [5] an attempt was made to study the amplitudes $S_n$ for small time-periodic perturbations. However, the appearance of resonant energies seems to be neglected in this paper.

2. The Hamiltonian $H_i$ corresponding to a zero-range potential well of a "depth" $h_i$ is defined as $H_i = -\frac{d^2}{dx^2}, \ x \in \mathbb{R}_+$, with the boundary condition $u'(0) = -h_i u(0)$, $h_i = h_i$. The operator $H_i > 0$, if $h_i \leq 0$, and it has (exactly one) negative eigenvalue $\lambda = -h_i^2$ with the eigenfunction $\exp(-h_i x)$, if $h_i > 0$. Let $H_0 = -\frac{d^2}{dx^2}$ with the boundary condition $u(0) = 0$ be the "free" Hamiltonian. The scattering matrix $S^{(1)}(\lambda)$ for the pair $H_0$, $H_i$ at energy $\lambda$ equals

$$S^{(1)}(\lambda) = (h_i - i \lambda^{1/2}) (h_i + i \lambda^{1/2})^{-1}.$$  

We shall consider zero-range potential well whose depth depends periodically on time. Mathematically this problem is governed by the equation

$$i \frac{\partial u}{\partial t} = -\frac{\partial^2 u}{\partial x^2}, \ x \in \mathbb{R}_+,$$  

with the time-dependent boundary condition

$$u'(0,t) = h(t) u(0,t), \ \overline{h(t)} = h(t), \ h(t+2\pi) = h(t)$$  

We will look for solutions of equation (1) which have a representation of the form

$$u(x,t) = \sum_{n = -\infty}^{\infty} u_n(x) e^{-i(n+\theta)t}$$
where the parameter \( \theta \in [0,1] \). Such solutions describe a stationary process in the sense that for any \( \tau \in \mathbb{R} \)

\[
(2\pi)^{-1} \int_{\tau}^{\tau+2\pi} |u(x,t)|^2 \, dt = \sum_{n=-\infty}^{\infty} |u_n(x)|^2
\]  

(5)

Substituting (4) into (2) we find that \( u_n(x) \) should satisfy the equations

\[-u_n''(x) = (n+\theta) u_n(x),\]

(6)

whose solutions are linear combinations of exponentials. In particular, the solution corresponding to the incoming wave \( \exp(-i\lambda^{1/2} x) \), \( \lambda = m+\theta, m \in \mathbb{Z}, \theta \in [0,1] \), has the form

\[u_n(x,\lambda) = S_{nm} \exp(-i\lambda^{1/2} x) - S_{n} \exp(i(\theta+n)^{1/2} x),\]

(7)

where \( S_{nm} = 1, S_{n} = 0 \), if \( n \neq m \), and

\[i(\theta+n)^{1/2} = -|\theta+n|^{1/2}, n \leq -1.\]

The terms \( S_{n} \exp(i(\theta+n)^{1/2} x) \) describe outgoing waves, if \( n \geq 0 \), and they are exponentially decaying, if \( n < 0 \).

Equations (6) are coupled by the boundary condition (3) which allows us to determine the amplitudes \( S_{n}(\lambda) \). In fact, substituting (7) into (4) and then into (3) we obtain the equation

\[-i\lambda^{1/2} e^{-int} - i \sum_{n=-\infty}^{\infty} (\theta+n)^{1/2} S_{n}(\lambda) e^{-int} = h(t) (e^{-int} - \sum_{n=-\infty}^{\infty} S_{n}(\lambda) e^{-int}).\]

(8)

Expanding \( h(t) \) in the Fourier series and comparing coefficients of \( e^{-int} \) we arrive at an infinite set of algebraic equations for the amplitudes \( S_{n}(\lambda) \).

Note that functions \( S_{n}(\lambda) \) are continuous in \( \lambda \in [m, m+1] \) for every \( m = 0,1,2,... \). Moreover, \( S_{n}(m-0) = S_{n+1}(m+0) \) for all \( n \in \mathbb{Z} \) and \( m = 1,2,...,\)

3. Below we restrict ourselves to the consideration of the simplest case

\[h(t) = -h_1 + 2\epsilon \cos t\]

(9)
Then equation (8) is equivalent to the following system of equations

\[(i(\theta+n)^{1/2} + h_1) S_n - \varepsilon (S_{n+1} + S_{n-1}) = S_n^{(\varepsilon)} , \ n \in \mathbb{Z}, \quad (10)\]

where

\[S_m^{(\varepsilon)}(\lambda) = h_1 - i \lambda^{1/2} , \ S_{m-1}^{(\varepsilon)}(\varepsilon) = S_{m+1}^{(\varepsilon)}(\varepsilon) = - \varepsilon \quad (11)\]

and \(S_n^{(\varepsilon)} = 0\) for \(|n-m| > 2\). We emphasize that the amplitudes \(S_n = S_n(\lambda,\varepsilon)\) depend on energy \(\lambda\) of incoming wave and on the parameter \(\varepsilon\) in (9). It is convenient to rewrite the system (10) in vector notation. Set \(s = \{S_n\}, \ s_0 = \{S_n^{(\varepsilon)}\}, \ n \in \mathbb{Z}, \) and

\[\Lambda = \text{diag} \{i(\theta+n)^{1/2} + h_1\}, \ K = \Gamma + \Gamma^*,\]

where \(\Gamma, (\Gamma 6)_n = 6_{n+1}\), is the shift operator. Then (10) is equivalent to the equation

\[(\Lambda - \varepsilon K) s = s_0 \quad (12)\]

which can be considered, for example, in the space \(l_2(\mathbb{Z})\).

In the case \(\varepsilon = 0\) the function (9) does not depend on \(t\) so that equations (10) become independent and can be easily solved. In fact, \(S_m(\lambda,0) = S^{(1)}(\lambda)\) and \(S_n(\lambda) = 0\), if \(n \neq m, n \geq 0\). For negative \(n\) the amplitude \(S_n(\lambda,0) = 0\) in case

\[h_1 = |\theta+n|^{1/2}\]

and \(S_n(\lambda,0)\) is arbitrary in case \(h_1 = |\theta+n|^{1/2}\). The latter equality is possible only if \(h_1 > 0\) and \(\lambda - \lambda_1 \in \mathbb{Z}\). In this case the function (4) is given by the relation

\[u(x,t) = \exp(-i\lambda^{1/2}x) - S^{(1)}(\lambda) \ exp(i\lambda^{1/2}x)) \ exp(i\lambda t) + \gamma \ exp(-h_1x + ih_1^2t) \quad (14)\]

with arbitrary \(\gamma\). The last term in (14) disappears (i.e. \(\gamma = 0\)) if \(h_1 \leq 0\) or \(h_1 > 0\) and \(\lambda - \lambda_1 \notin \mathbb{Z}\).

4. Our goal is to study the limit of the amplitudes \(S_n(\lambda,\varepsilon)\) as \(\varepsilon \to 0\). We first consider the non-resonant case when either \(h_1 \leq 0\) or \(h_1 > 0\) and \(\lambda - \lambda_1 \notin \mathbb{Z}\). Then condition (13) holds for all \(n = -1, -2, \ldots\) so that the operator \(\Lambda\) is invertible and (10) is equivalent to the relation

\[(I - \varepsilon \Lambda^{-1} K) s = \Lambda^{-1} s_0\]

Since \(K\) is a bounded operator, for sufficiently small \(\varepsilon\) this equation can be solved by
Thus for non-resonant energies $\lambda, \lambda - \lambda_1 \notin \mathbb{Z}$, the asymptotic expansion of amplitudes is described by regular perturbation theory. In particular, (15) ensures that $S_n(\lambda, \varepsilon) = O(\varepsilon^{n-m})$ so that the probability of excitation of states with energies $\lambda + K, K \in \mathbb{Z}$, is proportional to $\varepsilon^{\mid K\mid}$. The amplitude $S_m(\lambda, \varepsilon)$ converges to the scattering matrix (1), i.e.

$$S_m(\lambda, \varepsilon) = \frac{\varepsilon}{\lambda^{1/2}} (h_1 + i (\lambda + 1)^{1/2})^{-1} + O(\varepsilon^2).$$

The leading term of the corrections to the case $\varepsilon = 0$ is determined by the amplitudes

$$S_{m-1}(\lambda, \varepsilon) = -2i\varepsilon (h_1 + i (\lambda + 1)^{1/2})^{-1} (h_1 + i \lambda^{1/2})^{-1} + O(\varepsilon^2).$$

5. If $h_1 > 0$ and $\lambda$ equals one of the resonant points $\lambda_1 + K, K \in \mathbb{Z}$, there arises a non-trivial interaction of the incident wave with the quasi-bound state of the time-dependent well. This interaction does not vanish in the limit $\varepsilon \to 0$. From the mathematical viewpoint the problem is due to the appearance of zero eigenvalues of the operator $A$. The operator $A - \varepsilon K$ is invertible for all $\varepsilon > 0$ but some of the matrix elements of $(A - \varepsilon K)^{-1}$ tend to infinity as $\varepsilon \to 0$. For definiteness we suppose that $0 < h_1 < 1$ and $\lambda$ approaches the point $\lambda_0 = 1 - h_1^2$. In this case the resonant interaction is the most significant. In fact, we shall obtain asymptotic formulas for $S_n(\lambda, \varepsilon)$ which hold uniformly in $\lambda \in I_\delta = [\delta, 1-\delta], \delta > 0$, as $\varepsilon \to 0$.

To bypass the problem of small denominators which appears now we distinguish equation (10) with $n = -1$

$$\left( h_1 - (1-\lambda)^{1/2} \right) S_{-1} - \varepsilon (S_0 + S_{-2}) = -\varepsilon$$

where all coefficients vanish as $\lambda \to \lambda_0$ and $\varepsilon \to 0$. First we consider only equations in (10) which correspond to $n \geq 0$. We shall solve this system with respect to amplitudes $S_n, n \geq 0$, with $S_{-1}$ playing the role of a parameter. Since all diagonal elements $i(\lambda + n)^{1/2} + h_1, n \geq 0$, are separated from zero, this system can be solved by iteration which gives the relation

\[ iteration: \]

\[ s(\varepsilon) = \sum_{p=0}^{\infty} \varepsilon^p (\Lambda^{-1}K)^p \Lambda^{-1} s_0(\varepsilon). \]
We emphasize that quantities as $0(\varepsilon^2)$ are uniform in $\lambda \in I_\varepsilon$. Similarly, solving equations in (10) corresponding to $n \leq -2$ with respect to $S_{n}$, $n \leq -2$, we find that

$$S_{-2} = \varepsilon (\lambda_1 - (2-\lambda)^{1/2})^{-1} S_{-1} (1+0(\varepsilon^2)).$$

(20)

Substituting expressions (19), (20) into (18) we obtain finally the equation for $S_{-1}$. It follows that

$$S_{-1}(\lambda,\varepsilon) = 2i\varepsilon \lambda^{1/2} \Omega^{-1}(\lambda,\varepsilon) (1+0(\varepsilon)).$$

(21)

where

$$\Omega(\lambda,\varepsilon) = [-h^2 + (1-\lambda)^{1/2} + \varepsilon^2 (h_1 - (2-\lambda)^{1/2})^{-1}] (h_1 + i\lambda^{1/2}) + \varepsilon^2$$

Here we have taken into account that

$$|\varepsilon^2 \Omega^{-1}(\lambda,\varepsilon)| \leq C.$$

Combining (19) with (21), we find also the asymptotics of $S_0$:

$$S_0(\lambda,\varepsilon) = (h_1 - i\lambda^{1/2}) (h_1 + i\lambda^{1/2})^{-1} + 2i\varepsilon^2 \lambda^{1/2} (h_1 + i\lambda^{1/2})^{-1} \Omega^{-1}(\lambda,\varepsilon) + 0(\varepsilon).$$

(22)

Clearly, $|S_0(\lambda,\varepsilon)| = 1$ up to an error of order $\varepsilon$.

If $\lambda$ is separated from the point $\lambda_0$, we can replace $\Omega(\lambda,\varepsilon)$ by $\Omega(\lambda,0)$ which is not zero. In this case we recover the relations (16), (17) (for $m = 0$). In the particular case $\lambda = \lambda_0$ we have that

$$\lambda_0,\varepsilon) = \varepsilon^2 (h_1 - (1+h_1^{3/2})^{-1} b_1$$

where

$$b_1 = 2h_1 - (1+h_1^{3/2}) + i (1-h_1^{3/2})$$

There fore according to (21), (22)

$$S_{-1}(\lambda_0,\varepsilon) = 2i (1-h_1^{3/2}) (h_1 - (1+h_1^{3/2}) b_1^{-1} \varepsilon^{-1} + 0(1),
S_0(\lambda_0,\lambda) = \overline{b_1} b_1^{-1} + 0(\varepsilon).$$

As could be expected, the amplitude $S_{-1}(\lambda_0,\varepsilon)$ grows infinitely as $\varepsilon \to 0$. By virtue of (5) it follows that for the corresponding function (4) and any $r > 0$ the integral

$$\int_{-\infty}^{\infty} F(s) dr$$

tends to infinity as $\varepsilon \to 0$. This is consistent with the decoupling of bound states and
scattering states in the stationary case $\varepsilon = 0$ when, by (14), the integral (23) has arbitrary value.

The amplitude $S_0(\lambda_0, \varepsilon)$ has a finite limit $S_0(\lambda_0, 0)$ which is, however, different from the scattering matrix (1) at energy $\lambda_0$ for the time-independent boundary condition $u'(0) = -h_1 u(0)$. Therefore, at energy $\lambda_0$ we find an additional resonant phase shift which does not vanish in the limit $\varepsilon \to 0$.

6. In stationary problems resonances are usually defined as complex "eigenvalues" for which the Schrödinger equation has solutions satisfying the outgoing radiation condition at infinity. Similarly, a complex point $\lambda$ can be called [3] resonant point for the problem (2), (3) if there exists its solution of the form

$$u(x,t) = \sum_{n=-\infty}^{\infty} A_n \exp \left[ i(\lambda + n)^{1/2} x - i(n+\lambda)t \right]$$

It is easy to see that at such $\lambda$ the homogeneous system of equations

$$(i(\lambda + n)^{1/2} + h_1) A_n - \varepsilon (A_{n+1} + A_{n-1}) = 0$$

should have a non-trivial solution. This system can be studied by the method of section 5. In the case $0 < h_1 < 1$ there exist for sufficiently small $\varepsilon$ resonant points obeying the relation

$$\lambda = n - h_1^2 - 2 \varepsilon^2 h_1 ((1 + h_1^2)^{1/2} + i(1-h_1^2)^{1/2}) + O(\varepsilon^4)$$

where $n$ is an arbitrary integer. In the limit $\varepsilon \to 0$ these complex points approach real points differing from $\lambda = -h_1^2$ by some integer.

References


[3] V.N. OSTROVSKII, many-photon ionization, resonance scattering on a
