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A CLASS OF WEIGHTED FUNCTION SPACES,
AND INTERMEDIATE CACCIOPPOLI-SCHAUDER ESTIMATES

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1 - A THEOREM OF D. GILBARG AND L. HORMANDER

Consider the Dirichlet problem

\[ \begin{align*}
L u &= f \text{ in } \Omega, \\
u &= \varphi \text{ on } \partial \Omega,
\end{align*} \]

where \( \Omega \) is a bounded open subset of \( \mathbb{R}^N \), \( \partial \Omega \) its boundary, and \( L \) a linear second order uniformly elliptic differential operator with coefficients defined on \( \tilde{\Omega} \). The classical Caccioppoli-Schauder approach to (1) provides, under suitable regularity assumptions about \( \partial \Omega \) and the coefficients of \( L \), a priori bounds on norms

\[ \| u \|_{C^k,\delta(\Omega)}, \quad k = 2, 3, \ldots \quad \text{and} \quad \delta \in ]0, 1[, \]

this of course requires, to start with, the membership of \( f \) in \( C^{k-2,\delta}(\overline{\Omega}) \) and of \( \varphi \) in \( C^k,\delta(\partial \Omega) \).

What happens now if we weaken our assumption about \( \varphi \) by requiring that it belong to \( C^{k',\delta'}(\partial \Omega) \) for some \( k' = 0, 1, \ldots \) and some \( \delta' \in ]0, 1[ \) such that \( k' + \delta' < k + \delta \)? An answer to this question was given by Gilbarg and Hörmander [4] : they provided weighted \( C^{k,\delta} \) norm estimates for solutions of (1), the weight consisting of the \( \alpha \)-th power of the distance from \( \partial \Omega \) with \( \alpha = k + \delta - (k' + \delta') \). Note that, for what correspondingly concerns \( f \), the natural regularity requirement is now only that its weighted \( C^{k-2,\delta} \) norm be finite.

In order to illustrate the key point of [4] we introduce some notations. Letting
(under the convention that the dependence on \( x^0, r \) be depressed if \( x^0 = O, r = 1 \)), we define \( C^{k, \delta}(B^+_R) \) as the space of functions \( u = u(x), x \in B^+_R \), having finite norms

\[
|u|_{C^{k, \delta}(B^+_R)} = \sup_{S > 0} S^\alpha |u|_{C^{k, \delta}(B^+_R[S])}
\]

here, \( k = O, 1, \ldots, O < \delta \leq 1, \alpha \geq O, \) and \( B^+_R[S] = \{ x \in B^+_R \mid x_N > S \} \).

(When \( \alpha < O \) the right-hand side in the above definition of norm is finite only for \( u = 0 \)). Through direct investigation of Green's function for the Laplace operator in the upper half space Gilbarg and Hörmander proved the following result (Theorem 3.1 of their paper): let \( k = 2, 3, \ldots, O < \delta \leq 1, O \leq \alpha < k + \delta \) and \( k + \delta - \alpha \notin \mathbb{N} \); then there exists a constant \( C \) such that

\[
(2)_k \quad |u|_{C^{k, \delta}(B^+_R)} \leq C |f|_{C^{k-2\delta}(B^+_R)}
\]

whenever \( u \) is a function from \( C^{k, \delta}(B^+_R) \) which vanishes near \( S^+ \) and satisfies (in the pointwise sense)

\[
(3) \quad u \big|_{S^0} = 0, \quad \Delta u = f \text{ in } B^+.
\]

What we are going to describe in the present article is an alternative approach to (3), which yields a slightly more general result than the bounds \((2)_k\). Notice that the passage from \( \Delta \) to more general variable coefficient operators \( L \) can be achieved through a perturbation argument as in [4, prop. 4.3]; the case of nonvanishing Dirichlet data \( \varphi \) on \( S^0 \) can be handled through suitable extensions of the \( \varphi \)'s to the upper half space [4, lemma 2.3]; finally, partitions of unity and changes of variables near boundary points lead to the general setting of (1) [4, theorem 5.1]. This procedure exhibits rather delicate technical features, if one wants to adopt the "natural" generality for what concerns regularity assumptions about the coefficients of \( L \) as well as \( \partial \Omega \). The crux of the matter lies, however, within the study of (3).

2 - THE MAIN RESULTS OF THIS ARTICLE
We are going to deal with weak solutions to a problem such as

\[ u \mid_{S^0} = O, \Delta u = f + f^i \text{ in } B^+ \]

i.e., for some \( p \in ]1, \infty[ \),

\[ u \in H^{1,p}(B^+) \text{, } u \mid_{S^0} = O, \]

\[ \int_{B^+} u \varphi \, dx = \int_{B^+} (-f \varphi + f^i \varphi^i) \, dx \quad \forall \varphi \in C_0^{\infty}(B^+) \]

(summation convention of repeated indices). Here and throughout, \( H^{k,p} \) and \( H^{k,p}_0 \) are the standard notations for Sobolev spaces.

For our study of regularity we find it convenient to introduce new (norms and) function spaces. Namely, for \( 1 \leq p < \infty \), \( \alpha \in \mathbb{R} \) and \( O \leq \lambda \leq N + p \) let

\[ [u]_{L_p^\alpha(B_R^+)} \equiv \sup_{B_R^+ \ni x^0} \rho^{-\lambda} \inf_{c \in \mathbb{R}} \int_{B_R^+(x^0)} x_N^{\alpha} |u - c|^p \, dx \]

and denote by \( L_p^\alpha(B_R^+) \) the space of functions \( u = u(x), x \in B_R^+ \), having finite norms

\[ |u|_{L_p^\alpha(B_R^+)} = (\int_{B_R^+} x_N^{\alpha} |u|^p \, dx + [u]_{L_p^\alpha(B_R^+)}^{1/p}) \]

It is clear that, for any value of \( \alpha \), \( L_p^\alpha(B_R^+) \) at least contains \( C_0^{\infty}(B_R^+) \).

\( L_p^\alpha(B_R^+) \) is the by now classical campanato space, and \( L_p^\alpha(B_R^+ \sim C^{0,(\lambda - N)/p}(B_R^+) \) if \( N < \lambda \leq N + p \) [2]. But we have more:

**Lemma 1**

For \( \alpha \geq O \) and \( N < \lambda \leq N + p \) the spaces \( L_p^\alpha(B_R^+) \) and \( C_0^{0,(\lambda - N)/p}(B_R^+) \) are isomorphic.

\( L_p^{0,N}(B_R^+) \) is a \( B M O \) (\( = \) Bounded Mean Oscillation) space [6]. The importance of \( B M O \) spaces as "good substitutes" for \( C^0 \) and \( L_\infty \) has since long been acknowledged in PDE's (and Harmonic Analysis ...). Take for instance our initial considerations about the classical Caccioppoli-Schauder approach to (1):
$BMO$ spaces are known to fill the gaps left over by the exclusion of the two values $\delta = 0$ and $\delta = 1$ [3]. But weighted norms lead to another example. Precisely, consider the continuous imbedding

\[(6) \quad C_{\alpha + \beta}^{\delta + \beta}(B^+_R) \subseteq C_{\alpha}^{\delta}(B^+_R)\]

which is proven in [4] for $\alpha \geq 0$, $0 \leq \delta < 1$ and $\beta > 0$ with $\delta + \beta \leq 1$, under the restriction $\alpha \neq \delta$. This restriction has far-reaching consequences, such as the above-mentioned requirement $k + \delta - \alpha \in \mathbb{N}$ for the validity of (2). But, why cannot $\alpha = \delta$ be allowed? For sure, (6) is false when $\alpha = \delta = 0$, as the one-dimensional example given in [4], that is, $u(x) = \log x$, $0 < x < 1$, clearly shows. But, as it happens, this function $u$ belongs to $L_{0}^{p,N}(O,1)$... We can indeed prove the following result, which contains (6) in all cases except $\alpha \neq O = \delta$.

**Lemma 2**

For $\alpha \geq 0$, $0 \leq \delta < 1$ and $\beta > 0$ with $\delta + \beta \leq 1$, the continuous imbedding

\[L_{\alpha + \beta}^{\delta + \beta}(B^+_R) \subseteq L_{\alpha}^{p,N+p \delta}(B^+_R)\]

is valid.

We can now arrive at our results about solutions to (5). Adopting the symbol $L_{\alpha}^{\infty}(B^+_R)$ to denote the space of measurable functions $h = h(x)$, $x \in B^+$, such that

\[\left\| h \right\|_{L_{\alpha}^{\infty}(B^+_R)} = \left\| x_{\alpha}^{\beta} h \right\|_{L_{\alpha}^{\infty}(B^+_R)}\]

is finite, we begin with first derivatives.

**Theorem 1**

Let $0 \leq \delta < 1$, $0 \leq \alpha < 1 + \delta$. If, for a suitable value of $p > 1$, $u$ satisfies (5) with $f \in L_{1+\alpha-\delta}(B^+_R)$ and $f^1, ..., f^N \in C_{\alpha}^{\delta}(B^+_R)$, then all its first derivatives belong to $L_{\alpha}^{p,N+p \delta}(B^+_R)$, $0 < R < 1$, and satisfy

\[\sum_{i=1}^{N} \left\| u_i \right\|_{L_{\alpha}^{p,N+p \delta}(B^+_R)} \leq C \left( \left\| f \right\|_{L_{1+\alpha-\delta}(B^+_R)} \right) + \sum_{i=1}^{N} \left| f_i \right|_{C_{\alpha}^{\delta}(B^+_R)} + \left| u \right|_{H_{1+\delta}(B^+_R)}\]

with $C$ independent of $u, f, f^1, ..., f^N$. 

IX-4
The passage to second derivatives is performed, so to speak, through "differentiation" of (5) with respect to $x_1, \ldots, x_{N-1}$. Without loss of generality, it can be assumed that $f^1 = \ldots = f^N = O$; as for $f$, the "natural" requirement becomes

$$f \in C_0^0, \delta (B^+).$$

for $O \leq \alpha < 2 + \delta$. It is the range $1 + \delta \leq \alpha < 2 + \delta$, of course, that poses new difficulties: no longer is then $f$ in some $L^p (B^+)$, so that the $H^{2,p}$ regularity theory does apply to (5), and the above results about $u$ are not inherited by $u_{x_S}, S = 1, \ldots, N-1$. But $H^{2,p}$ regularity does apply to $x_N u$, and $U = x_N u_{x_S}$ satisfies, in the weak sense,

$$U \bigg|_{B^+_R} = O, \quad \Delta U = -x_N f_{x_S} + 2 u_{x_S} x_N \text{ in } B^+_R$$

for any $R_1 < |O|, 1|$. We can thus arrive at.

**Theorem 2**

Let $0 < \delta < 1$, $0 \leq \alpha < 2 + \delta$. If, for a suitable value of $p > 1$, $u$ satisfies (5) with $f \in C_0^0, \delta (B^+)$ and $f^1 = \ldots = f^N = O$, then all its second derivatives belong to $L^p_{\alpha, \alpha + \delta} (B^+_R)$ when restricted to $B^+_R$, $O < R < 1$, and satisfy

$$\sum_{i,j=1}^{N} |u_{x_i x_j}|_{L^p_{\alpha, \alpha + \delta} (B^+_R)} \leq C (|f|_{C_0^0, \delta (B^+)} + |u|_{H^1, p (B^+)})$$

with $C$ independent of $u, f$.

(If we want to be more specific in the choice of $p$, we take $p = 2$ for $O \leq \alpha < \frac{1}{2} + \delta$ and $1 < p < \frac{1}{\alpha - \delta}$ for $\frac{1}{2} + \delta \leq \alpha < 1 + \delta$ in both Theorems 1 and 2, $p = 2$ for $1 + \delta \leq \alpha < \frac{3}{2} + \delta$ and $1 < p < \frac{1}{\alpha - 1 - \delta}$ for $\frac{3}{2} + \delta \leq \alpha < 2 + \delta$ in Theorem 2).

When $\text{supp } u \cap S^+ = \emptyset$, (7) holds for $R = 1$ without the term $|u|_{H^{1,p} (B^+)}$ on its right hand side. This means that (2)_2 holds for all values of $\alpha$ in the range $[O, 2 + \delta[, O < \delta < 1$, that is, without exception for $\alpha = \delta$ and $\alpha = 1 + \delta$. Since the procedure leading to Theorem 2 can be repeated for all higher order derivatives, (2) holds whenever $k = 2, 3, \ldots$ and $O \leq \alpha < k + \delta$, $O < \delta < 1$, no exception being made for $k + \delta - \alpha \in \mathbb{N}$.

As for $\delta = O$, we simply mention that $C_0^0, 0 (B^+)$ could safely be
replaced by $L^\infty_\alpha (B^+)$ throughout. The above results can therefore be said to contain "weighted versions of the $L^\infty \to BMO$ type of regularity".

A few words about our techniques. The main tools are estimates such as

$$\int_{B_{\rho}(x^0)} |\nabla w|^p \, dx \leq C(p) \left[ \frac{\rho}{r} \right]^N \int_{B_{\rho}(x^0)} |\nabla w|^p \, dx + \sum_{i=1}^{N} \int_{B_{\rho}(x^0)} |h^i|^p \, dx$$

and

$$\int_{B_{\rho}(x^0)} |\nabla w - (\nabla w)_{\rho;\alpha}|^p \, dx \leq C(p,\alpha) \left[ \frac{\rho}{r} \right]^{Np} \int_{B_{\rho}(x^0)} |\nabla w - (\nabla w)_{r\alpha}|^p \, dx$$

$$+ \sum_{i=1}^{N} \int_{B_{\rho}(x^0)} |h^i - (h^i)_{r\alpha}|^p \, dx,$$

which hold whenever $w$ satisfies

$$w \in H^{1,p}(B_r(x^0)), \quad \int_{B_{\rho}(x^0)} w_{x_i} \varphi_{x_i} \, dx = \int_{B_{\rho}(x^0)} h^i \varphi_{x_i} \, dx \quad \forall \varphi \in C_0^\infty(B_r(x^0))$$

where $0 < \rho \leq r < \infty$, $x^0 \in \mathbb{R}^N$; in (9), the symbol $(.)_{\rho;\alpha}$ denotes average over $B_{\rho}(x^0)$ with respect to $x^\alpha_N \, dx$, $\alpha \geq O$. We need $p$ from $[1,2]$. For $p = 2$, (8) and (9) are obtained [3] through typical techniques of the Hilbert space theory of elliptic PDE's. The passage to $1 < p < 2$ requires some preliminary results from the corresponding $H^{k,p}$ theory which can be found, for instance, in [7].

If spheres $B_{\rho}(x^0)$ are replaced throughout by hemispheres $B_{\rho}^+(x^0)$ - and $w$ is required to vanish on $S_r^0(x^0)$ - the counterpart of (8) is obviously valid for $1 < p \leq 2$, while the counterpart of (9) is only needed here for $p = 2$ as in [3].

Detailed proofs will appear in a forthcoming article.
The results mentioned here could be compared with those of [1], [5], where the perturbing role of the boundary appears through degeneration of operators rather than explosion of some norms of free terms (and boundary data).

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