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THE NASH-MOSER INVERSE MAPPING THEOREM

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To prove a local embedding theorem for strongly pseudo-convex CR structures (of dimension \(m\)) (cf. [2]) we used a variant of Nash-Moser inverse mapping theorem. We try to explain in general terms how it was done, without bothering too much about technical details.

For a \(\varepsilon_0 > 0\) we define \(\varepsilon_n > 0\) inductively by

\[
\varepsilon_{n+1} = \varepsilon_n^a, \quad a = 3/2.
\]

The Nash-Moser inverse mapping theorem (cf. [3]) is based on the following:

**Lemma:**

Let \(s, t > 0\) be given. Pick \(\lambda, \mu > 0\) so large that

\[
s + (a-2) \leq 0
\]

\[
t + a^2 \mu + (1-a)\lambda \leq -a.
\]

Let \(p_n > 0\) be a sequence. Assume that for a constant \(C^* > 0\)

\[
p_n \leq C^* (\varepsilon_n^{s} (p_{n-1})^2 + \varepsilon_n^{\lambda} \varepsilon_{n-1}^{\lambda-t})
\]

\[
\varepsilon_1 \leq 1/(2C^*)^2, \quad p_0 \leq \varepsilon_1^{\mu}/2C^*.
\]

Then

\[
p_n \leq (\varepsilon_{n+1})^{\mu}/2 C^*
\]

Proof goes as follows: we set \(g_n = \varepsilon_{n+1}^{-\mu} p_n\). Then

\[
g_n \leq C^* (\varepsilon_n^{s} (s+(a-2)\mu) g_{n-1}^2 + \varepsilon_{n-1}^{-(t+a^2 \mu + (1-a)\lambda)}).
\]
Hence $g_{ij} \leq C^* \left( (g_{ij})^2 + \varepsilon_{ij} \right)$. We now prove $g_{ij} \leq 1/2 C^*$ by induction on $\nu$.

We apply the above lemma in the following setting: we consider open sets $F', G'$ in Frechet spaces $F, G$ and a map

$\phi : F' \to G'$

Each of these Frechet spaces is assumed to be endowed with an increasing sequence of semi-norms $|| \cdot ||_k$ which defines its topology. In practice, we consider the Frechet spaces of $C^\infty$ sections of vector bundles over a manifold $M$. $|| \cdot ||_k$ is defined by measuring the partial derivatives up to degree $k$ of sections. $\phi$ is given by a non-linear partial differential operator involving partial derivatives up to order, say $r$. This is translated into an assumption

\begin{equation}
||\phi(f)||_k \leq C_k (1 + ||f||_{k+r})
\end{equation}

For $k$ sufficiently large any map with the above assumption is called tame (cf. R. Hamilton [1] for more details). We assume that $\phi$ is infinitely differentiable and all partial derivatives are tame. In particular there is for each $f \in F'$ a continuous linear map.

$d_f \phi : F \to G$

such that with $R_f(h) = \phi(f+h) - \phi(f) - d_f \phi(h)$

\begin{equation}
||R_f(h)||_k \leq C_k (||f||_{k+r} ||h||_{k+o} + ||h||_{n+r})
\end{equation}

for $k \geq k_f$. We also assume that there is a mollifier $M_\varepsilon$ ($\varepsilon > 0$) with the standard properties: for $s \geq 0$

\begin{equation}
||M_\varepsilon f||_{k+s} \leq C_{k,s} \varepsilon^{-s} ||f||_k
\end{equation}

\begin{equation}
||f - M_\varepsilon f||_k \leq C_{k,s} \varepsilon^s ||f||_{k+s}
\end{equation}
We now wish to show that an element \( g \in \text{im} \phi \) is in the image of \( \phi \). We may assume that \( g = 0 \). We solve the problem by a successive approximation. Namely, for the \( a \)-th approximation \( f_a \) we define \( f_{a+1} \) as follows: note that \( \phi(f_a + h) \) is very closed to \( \phi(f_a) + d_{f_a} \phi(h) \). Hence we solve the equation:

\[
(8) \quad \phi(f_a) + d_{f_a} \phi(h) = 0
\]

However, in the process we usually lose derivatives. We compensate this by setting

\[
(9) \quad f_{a+1} = f_a + M_{\varepsilon_{a+1}} h_a
\]

where \( h_a \) is a solution of (8) and where \( \varepsilon_a \) is given in (1). In fact, we assume that we can find \( h_a \) with

\[
(10) \quad \|h_a\|_{k-r} \leq C_k \|\phi(f_a)\|_k
\]

This estimate is essential for this method to work. In order to show that \( f_a \) converge to a solution \( f \) of our problem, it is enough to show that \( p_a = \|\phi(f_a)\|_k \) satisfy (3) in the lemma. If this is the case, \( p_a \) has estimate (4). In view of (9) and (10) it then follows that \( f_a \) will also converge. Now:

\[
\phi(f_a + M_{\varepsilon_{a+1}} h_a) = \phi(f_a) + d_{f_a} \phi(M_{\varepsilon_{a+1}} h_a) + R_{f_a} (M_{\varepsilon_{a+1}} h_a)
\]

\[
= R_{f_a} (M_{\varepsilon_{a+1}} h_a) - d_{f_a} \phi(h_a) - M_{\varepsilon_{a+1}} h_a
\]

Note (7) and (6). From the first term (resp. the second term) we obtain terms \( C^s_{\varepsilon_{a+1}} -s(p_a)^2 \) (resp. \( \varepsilon_{a+1}^{-\lambda-t} \)) for a choice of \( s \) and \( t \).

The above shows that we can solve the equation \( \phi(f) = g \) for a given \( g \) provided we find a very good approximation \( f_o \) so that the last inequality in (30) is satisfied. In particular, we find that a small neighborhood of \( f_o \) is covered by \( \phi \).

For a local embedding theorem mentioned in the beginning we have a following more general setting. Namely, we have a manifold \( M \) and for each open \( U \subseteq M \) we have:
with $\Psi, \Phi = 0$. They are related by compatible restriction maps. We are given $g \in G'\!(M)$ with $\Psi(g) = 0$ and a reference point $p_o$ in $M$. We wish to show that the restriction of $g$ to a suitable open neighborhood $\mathcal{U}$ of $p_o$ is in the image of $\Phi$. We may assume that $g = 0$. The existence of $\Psi$ means that we may not be able to solve the equation (8). We have to replace $\Phi(f_o)$ by its projection to the image of $d_f \phi$. Moreover, we could find such projection only for $\mathcal{U}$ satisfying certain conditions which also depend on $f_o$. Namely, for each $f \in F(U_1)$, where $p_o \in U_1$, we have a way to define $r_f > 0$ and a distance function $t_f$ to $p_o$ with the following properties: for $0 < r < r_f$ set

$$
(11) \quad \mathcal{U}(f, r) = \{ p \in U_1 ; t_f(p) < r \}.
$$

Let $f'$ be the restriction of $f$ to $F(U(f, r))$, $f' = f|_{U(f, r)}$. Then there is

$$
(12) \quad V_f' : G(U(f, r)) \to F(U(f, r))
$$
such that with $h' = V_f' (\Phi(f'))$

$$
(13) \quad -\Phi(f') = (d_f \phi)(h') + A(\Phi(f'))
$$

where $A(\Psi)$ is given by a composition.

$$
(14) \quad A(\Psi) = A_1 \circ A_2(\Psi)
$$

$A_1$ is a linear map. $A_2$ is a non-linear partial differential operator starting with quadratic terms. Since our error term $A(\Phi(f'))$ is of quadratic nature as $R_f$ in (6) we may try to solve our problem by the same method as in the standard case.

We first find $f_o \in \mathcal{F}(U_o)$ such that:

$$
(15) \quad \| \Phi(f_o) \|_{\mathcal{U}(f_o, r)} \|_k \leq O(r^N)
$$
for all $N$. This is achieved by solving the differential equation $\phi(f) = 0$ as a formal power series centered at $p_o$ whose Taylor series agree with the solution formal power series will satisfy our requirement. For $\alpha > \beta$ we set for $0 < r_o < r_f$

$$
(16) \quad \varepsilon_o = r_o^\alpha, \quad \delta_o = r_o^\beta
$$

and define $\varepsilon_v$ and $\delta_v$ as in (1). We then set

$$
(17) \quad r_{v+1} = r_v - 3 \delta_v
$$

Starting from $f_o|U(f_o, r_o)$ we construct $f_1$ as in the standard case replacing $h$ in (8) by $h'$ in (13). We then show that, if $r_f$ is properly chosen, $r_1 + \delta_1 < r_f$ and $U(f_1, r_1 + \delta_1) \subseteq U(f_o, r_o - \delta_o)$. We then consider $f_1|U(f_1, r_1)$ and proceed inductively. We do this construction for all $r_o$ in $]0, r_f[$.

Thus the success of our method depends essentially on the nature of $V_{f', v}$ in (12) which solves the equations (13) and how we could handle the new error term $A(\phi(f'))$. In our case $V_{f'}$ is obtained by using the solution mapping $N_{f'}$ of generalized Neumann boundary value problem on $U(f, r)$ associated with $d_{f'}$. $N_f$ also enters in the construction of $A_1$ in (14). The fact is we could only find $N_f$ for $U(f, r)$ as in (11), where $t_f$ satisfies certain conditions. This is the reason why we had to change $U$ as each step of the successive approximation. On the other hand, since $U(f_{v+1}, r_{v+1}) \subseteq U(f_v, r_v)$, we could use the interior estimate. In such estimate a factor $(\varepsilon_v)^{-\ell}$ (cf. (17)) will come in the constant of the inequality. However, we can admit such factor in view of (3). Using estimates for $N_f$ on $U(f, r)$ as well as interior estimate, we prove the first inequality in (3) for all $r$ in $]0, r_f[$, provided $p_o, \ldots, p_{v-1}$ is sufficiently small. We now need the second and the third inequality in (3). In view of (16) the second is satisfied for sufficiently small $r_o$. Similarly the third is satisfied in view of (15).
REFERENCES:

